Krishna's

TEXT BOOK on

Differential Equations & Vector Calculus

As per Allahabad State University Allahabad (U.P.)

(*w.e.f.* 2018-2019)

By

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Jai Shri Radhey Shyam

Dedicated
to
Lord
Krishna

Authors & Publishers

Preface

This book on **Differential Equations & Vector Calculus** has been specially written according to the latest **Allahabad State University Syllabus** to meet the requirements of the **B.A. and B.Sc. Part-II Students** of Allahabad State University in Uttar Pradesh.

The subject matter has been discussed in such a simple way that the students will find no difficulty to understand it. The proofs of various theorems and examples have been given with minute details. Each chapter of this book contains complete theory and a fairly large number of solved examples. Sufficient problems have also been selected from various university examination papers. At the end of each chapter an exercise containing objective questions has been given.

We have tried our best to keep the book free from misprints. The authors shall be grateful to the readers who point out errors and omissions which, inspite of all care, might have been there.

The authors, in general, hope that the present book will be warmly received by the students and teachers. We shall indeed be very thankful to our colleagues for their recommending this book to their students.

The authors wish to express their thanks to **Mr. S.K. Rastogi**, *M.D.,* **Mr. Sugam Rastogi**, *Executive Director,* **Mrs. Kanupriya Rastogi**, *Director* and **entire team** of **KRISHNA Prakashan Media (P) Ltd.**, **Meerut** for bringing out this book in the present nice form.

The authors will feel amply rewarded if the book serves the purpose for which it is meant. Suggestions for the improvement of the book are always welcome.

— Authors

Syllabus

Differential Equations & Vector Calculus

As per Allahabad State University
(*w.e.f.* 2018-19)

B.A./B.Sc. Paper-II M.M. : 50

Unit-1: Differential equation, formation of a differential equation, order and degree, methods of solution of a differential equation of first order and first degree, method of separation of variables, solution of homogeneous equation, linear equation and exact equations.

Unit-2: Linear differential equations with constant, coefficients, homogeneous linear differential equations. Differential equations of the first order but not of the first degree. Clairaut's equations and singular solutions, orthogonal trajectories.

Unit-3: Simultaneous linear differential equations with constant coefficients, general linear differential equations of the second order, method of solution in terms of one unknown integral belonging to the complementary function. Method of solution by transformation of the equation to the normal form, transformation of the equation by changing the independent variable. Method of operational factors, method of variation of parameters.

Unit-4: Vector differentiation and integration, gradient, divergence and curl, and their properties.

Unit-5: Line integrals, surface and volume integrals, Gauss, Green and Stokes theorems and problems based on these.

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DIFFERENTIAL EQUATIONS AND VECTOR CALCULUS

Chapters

1. Differential Equations of First Order and First Degree
2. Differential Equations of The First Order But Not of The First Degree
3. Orthogonal Trajectories
4. Linear Differential Equations with Constant Coefficients

5. Homogeneous Linear Differential Equations (Cauchy-Euler Equations)

6. Ordinary Simultaneous Differential Equations

7. Linear Equations of Second Order with Variable Coefficients

8. Differentiation of Vectors

9. Gradient, Divergence and Curl

10. Integration of Vectors

11. Line Integrals

12. Green's, Gauss's and Stoke's Theorems

Chapter

1

Differential Equations of First Order and First Degree

1 Definitions

A **differential equation** is an equation containing the dependent and independent variables and different derivatives of the dependent variables w.r.t. one or more independent variables.

The **order of a differential equation** is the order of the highest derivative (or differential coefficient) occurring in the equation.

(Lucknow 2007; Meerut 09B, 10B; Bundelkhand 10)

The **degree of a differential equation** is the degree of the highest derivative (or diff. coeff.) which occurs in it, after the differential equation has been rationalized (*i.e.*, made free from radicals and fractions so far as derivatives are concerned). A differential equation is called **ordinary**, if the unknown function depends on only one argument (independent variable).

(Lucknow 2007; Meerut 09B, 10B; Bundelkhand 10)

A differential equation is said to be **partial** if there are *two or more* independent variables.

A differential equation is said to be **linear** if the dependent variable, say, 'y' and all its derivatives occur in the first degree, otherwise it is **non-linear**.

A function $y = f(x)$ is called a **Solution** (*or the primitive*) of a differential equation if, when substituted into the equation, it reduces the equation to an identity and the process of finding all the solutions is called integrating (*or solving*) *the differential equation*.

General solution: **(Lucknow 2007; Gorakhpur 09)**

A solution of a differential equation, containing independent arbitrary constants equal in number to the order of the differential equation is called its *general solution*.

Particular solution: **(Lucknow 2007)**

A solution obtained by giving particular values to the arbitrary constants in the general solution is called a *particular solution* or *particular integral*.

Arbitrary Constants: The solution of a differential equation may contain as many arbitrary constants as is the order of the differential equation *i.e.*, the solution of an *n*th order differential equation may contain *n* arbitrary constants.

Illustrative Examples

Example 1: *Find the differential equation of the family of curves* $y = Ae^x + (B/e^x)$*, for different values of A and B.* **(Kanpur 2015)**

Solution: We have $y = Ae^x + Be^{-x}$. ...(1)

To obtain the required differential equation the constants A and B are to be eliminated with the help of the given equation (1) and the two equations obtained by differentiating (1) once and twice. Thus differentiating (1), we get

$$\frac{dy}{dx} = Ae^x - Be^{-x}. \qquad \text{...(2)}$$

Now differentiating (2), we get

$$\frac{d^2y}{dx^2} = Ae^x + Be^{-x}. \qquad \text{...(3)}$$

Eliminating A and B between (1), (2) and (3), we obtain $\frac{d^2y}{dx^2} = y$, which is the required differential equation.

Example 2: *Find the differential equation of all circles of radius a,* **or** *By the elimination of the constants h and k, find the differential equation of which* $(x-h)^2 + (y-k)^2 = a^2$*, is a solution.*

Solution: The equation of all circles of radius a is given by

$$(x-h)^2 + (y-k)^2 = a^2, \qquad \text{...(1)}$$

h and k being parameters (*i.e.*, arbitrary constants).

Differentiating (1), we get

$$2\,(x-h)+2\,(y-k)\,\frac{dy}{dx}=0. \qquad \ldots(2)$$

Differentiating (2), we get

$$1+(y-k)\,\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0. \qquad \ldots(3)$$

From (2) and (3), we obtain

$$(x-h)=-\,(y-k)\,(dy/dx) \text{ and } \quad (y-k)=-\,\frac{[1+(dy/dx)^2]}{d^2y/dx^2}.$$

Substituting these values in (1) and simplifying, we obtain

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2,$$

which is the required differential equation.

Comprehensive Exercise 1

1. Find the differential equation of the family of curves

$$y=Ae^{2x}+Be^{-2x},$$

for different values of A and B. **(Bundelkhand 2003)**

2. Find the differential equation corresponding to

$$y=ae^{2x}+be^{-3x}+ce^{x},$$

where a,b,c are arbitrary constants.

3. Find the differential equation of the family of curves

$$y=e^{x}\,(A\cos x+B\sin x).$$

where A and B are arbitrary constants.

4. By eliminating the constants a and b obtain the differential equation of which $xy=ae^{x}+be^{-x}+x^2$ is a solution. **(Purvanchal 2014)**

5. Find the differential equation corresponding to the family of curves $y=c\,(x-c)^2$, where c is an arbitrary constant.

6. Show that $Ax^2+By^2=1$ is the solution of

$$x\left[y\,\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-y\,\frac{dy}{dx}=0.$$

7. Show that $v=(a/r)+B$ is a solution of

$$\frac{d^2v}{dr^2}+\frac{2}{r}\frac{dv}{dr}=0.$$

Answers 1

1. $\dfrac{d^2y}{dx^2}=4y$

2. $\dfrac{d^3y}{dx^3}-7\dfrac{dy}{dx}+6y=0$

3. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+2y=0$

4. $x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}-xy+x^2-2=0$

5. $8y^2=4xy\dfrac{dy}{dx}-\left(\dfrac{dy}{dx}\right)^3$

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2 Differential Equations of First Order and First Degree

The differential equations of first order and of first degree can always be written in the form

$$\boldsymbol{M + N\,(dy/dx) = 0 \quad \text{or} \quad M\,dx + N\,dy = 0,}$$

where M and N are some functions of x and y or are constants.

Here the differential equation being of 1st order, its general solution (or primitive) will contain only one arbitrary constant. All differential equations of the first order cannot be always solved. However they can be solved by suitable methods if they belong to any one of the following standard forms :

(a) Variables separable,

(b) Homogeneous equations,

(c) Equations of the form $\dfrac{dy}{dx}=\dfrac{ax+by+c}{a_1x+b_1y+c_1}$,

(d) Linear equations,

(e) Equations reducible to the linear form,

(f) Exact differential equations,

(g) Equations reducible to exact form,

(h) Equations which by suitable change of variable can be transformed to any of the above forms.

2.1 Variables Separable

If a differential equation of the first order and of the first degree is of the form

$$f_1(x)\,dx = f_2(y)\,dy, \qquad \ldots(1)$$

where $f_1(x)$ is a function of x *only* and $f_2(y)$ is a function of y *only*, then we say that the *variables are separable* in the differential equation.

In such equations it is possible to get dx and all the terms involving x on one side and dy along with all the terms involving y on the other side.

To solve such a differential equation *integrate the two sides* and add an arbitrary constant of integration to any one of the two sides.

Thus, integrating both the sides of (1), we get its solution as

$$\int f_1(x)\,dx = \int f_2(y)\,dy + c,$$

where c is an arbitrary constant. The arbitrary constant can be chosen in any form suitable for the answer, *i.e.*, we can replace it by $\log c$, $\tan^{-1} c$, $\sin c$, e^c, etc.

Illustrative Examples

Example 3: *Solve* $(1+x^2)\,dy = (1+y^2)\,dx$ **(Meerut 2003)**

Solution: The given equation can be written as $dy/(1+y^2) = dx/(1+x^2)$, in which the variables have been separated.

∴ integrating, we get $\tan^{-1} y = \tan^{-1} x + \tan^{-1} c$, where c is an arbitrary constant

or $\quad \tan^{-1} y - \tan^{-1} x = \tan^{-1} c$

or $\quad \tan^{-1} \dfrac{y-x}{1+yx} = \tan^{-1} c$, $\quad$ or $\quad \dfrac{y-x}{1+yx} = c$

or $\quad (y-x) = c\,(1+yx)$, which is the required solution.

Example 4: *Solve* $(1+e^x)\,y\,dy = (y+1)\,e^x\,dx.$

Solution: Here $\dfrac{y}{y+1}\,dy = \dfrac{e^x}{1+e^x}\,dx$, (the variables being separated)

or $\quad \left[1 - \dfrac{1}{y+1}\right] dy = \dfrac{e^x}{1+e^x}\,dx.$

∴ integrating, we get

$y - \log(y+1) = \log(1+e^x) + \log c$, ($c$ being an arbitrary constant)

or $\quad y = \log[c\,(y+1)(1+e^x)]$

or $\quad c\,(y+1)(1+e^x) = e^y$, which is the required solution.

Example 5: *Solve* $dy/dx = (x+y)^2.$ **(Avadh 2011; Kanpur 15)**

Solution: Here the variables are not separable but some suitable substitution will reduce the differential equation to a form in which the variables are separable. Here we put $x + y = v$.

Differentiating both sides w.r.t., 'x', we have

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dv}{dx} - 1.$$

By these substitutions the given equation reduces to

$$\frac{dv}{dx} - 1 = v^2, \quad \text{or} \quad \frac{dv}{dx} = v^2 + 1, \quad \text{or} \quad \frac{dv}{v^2+1} = dx.$$

Thus the variables being separated and so integrating, we get

$$\int \frac{dv}{v^2+1} = \int dx + c, \qquad \text{or} \qquad \tan^{-1} v = x + c$$

or $$\tan^{-1}(x+y) = x + c, \qquad [\because v = x + y]$$

or $x + y = \tan(x + c)$ as the required solution where c is an arbitrary constant.

Example 6: *Solve* $dy/dx = \sin(x+y) + \cos(x+y)$.

Solution: Let $x + y = v$. Then differentiating, we have

$$1 + \frac{dy}{dx} = \frac{dv}{dx}, \qquad \text{or} \qquad \frac{dy}{dx} = \frac{dv}{dx} - 1.$$

Substituting these values in the given equation, we get

$$\frac{dv}{dx} - 1 = \sin v + \cos v, \qquad \text{or} \qquad \frac{dv}{dx} = 1 + \sin v + \cos v$$

or $$\frac{dv}{(1+\cos v) + \sin v} = dx, \text{ separating the variables}$$

or $$\frac{dv}{2\cos^2 \frac{1}{2} v + 2 \sin \frac{1}{2} v \cos \frac{1}{2} v} = dx \qquad \text{or} \qquad \frac{\frac{1}{2}\sec^2 \frac{1}{2} v}{1 + \tan \frac{1}{2} v}\, dv = dx.$$

Now integrating, we get $\log(1 + \tan \frac{1}{2} v) = x + c$

or $$\log\{1 + \tan \tfrac{1}{2}(x+y)\} = x + c,$$

which is the required solution containing an arbitrary constant c.

Comprehensive Exercise 2

Solve the following differential equations :

1. **(i)** $(1+x)\, y\, dx + (1-y)\, x\, dy = 0$. **(Garhwal 2011)**

(ii) $(1-x^2)(1-y)\, dx = xy(1+y)\, dy$.

2. **(i)** $x^2(y+1)\, dx + y^2(x-1)\, dy = 0$.

(ii) $\left(y - x\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$. **(Purvanchal 2006; Avadh 10; Gorakhpur 08, 11)**

3. **(i)** $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$. **(Garhwal 2003; Agra 06; Meerut 09B)**

(ii) $dy/dx = e^{x-y} + x^2 e^{-y}$. **(Agra 2005; Avadh 07; Meerut 09; Purvanchal 10, 11; Rohilkhand 10; Bundelkhand 04; Kumaun 14)**

4. **(i)** $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$. **(Rohilkhand 2010)**

(ii) $3e^x \tan y\, dx + (1-e^x)\sec^2 y\, dy = 0$. **(Meerut 2008; Bundelkhand 04; Kumaun 14)**

5. **(i)** $(ds/dx) + x^2 = x^2\, e^{3s}$.

(ii) $(dy/dx) \tan y = \sin(x + y) + \sin(x - y)$. **(Garhwal 2002; Kumaun 13)**

6. **(i)** $\log(dy/dx) = ax + by$.

(ii) $xy\,(dy/dx) = (1 + y^2)(1 + x + x^2)/(1 + x^2)$.

7. **(i)** If $dy/dx = e^{x+y}$ and it is given that for $x = 1$, $y = 1$, find y when $x = -1$.

(ii) $(x - y)^2\,(dy/dx) = a^2$.

8. **(i)** $\cos(x + y)\, dy = dx$. **(Lucknow 2005; Gorakhpur 09; Purvanchal 10; Avadh 14)**

(ii) $\cos(x + y)\, dx = dy$. **(Gorakhpur 2005)**

9. **(i)** $\sin^{-1}(dy/dx) = x + y$.

(ii) $(dy/dx) + 1 = e^{x+y}$. **(Kumaun 2006)**

10. **(i)** $dy/dx = (4x + y + 1)^2$. **(Lucknow 2006)**

(ii) $\left(\dfrac{x + y - a}{x + y - b}\right)\dfrac{dy}{dx} = \left(\dfrac{x + y + a}{x + y + b}\right)$.

11. $(y - x)(dy/dx) = a^2$. **(Meerut 2006B)**

Answers 2

1. **(i)** $xy = ce^{y-x}$

(ii) $\log\{x(1 - y)^2\} = \frac{1}{2}x^2 - \frac{1}{2}y^2 - 2y + c$

2. **(i)** $\frac{1}{2}(x^2 + y^2) + (x - y) + \log\{c\,(x - 1)(y + 1)\} = 0$

(ii) $y = c\,(a + x)(1 - ay)$

3. **(i)** $\tan x \tan y = c$ **(ii)** $e^y = e^x + \frac{1}{3}x^3 + c$

4. **(i)** $\sin x\,(e^y + 1) = c$ **(ii)** $\tan y = c\,(1 - e^x)^3$

5. **(i)** $(e^{3s} - 1) = c_1\, e^{(3s + x^3)}$, where $c_1 = e^{3c}$ **(ii)** $\sec y = c - 2\cos x$

6. **(i)** $-(1/b)\,e^{-by} = (1/a)\,e^{ax} + c$

(ii) $\frac{1}{2}\log(1 + y^2) = \log x + \tan^{-1} x + \log c$

7. **(i)** $y = -1$ **(ii)** $y + c = \frac{a}{2}\log\left\{\frac{x - y - a}{x - y + a}\right\}$

8. **(i)** $y = c + \tan\frac{1}{2}(x + y)$ **(ii)** $\tan\frac{1}{2}(x + y) = x + c$

9. **(i)** $[-2/(x + c)] = 1 + \tan\frac{1}{2}(x + y)]$ **(ii)** $(x + c)\,e^{(x+y)} + 1 = 0$

10. **(i)** $4x + y + 1 = 2\tan(2x + k)$

(ii) $(b - a)\log\{(x + y)^2 - ab\} = 2(x - y + c)$

11. $y + a^2 \log(a^2 + x - y) = c$

2.2 Homogeneous Equations

Definition: A differential equation of the form $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$, where $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of x and y of the same degree, is called a *homogeneous equations*. **(Bundelkhand 2005)**

To solve such an equation, put

$$\boldsymbol{y = vx\ ;\ \text{whence}\ \frac{dy}{dx} = v + x\frac{dv}{dx}}.$$

These substitutions transform the given equation into an equation of the form

$$v + x\frac{dv}{dx} = f(v) \quad i.e., \quad x\frac{dv}{dx} = f(v) - v.$$

The variables are now separable. Separating the variables and integrating, we get $\int \frac{dv}{f(v) - v} = \log x + c$, where c is an arbitrary constant.

Now replacing v by (y/x) after integration, we get the required solution.

Working Rule: To get the solution of a homogeneous differential equation proceed as follows :

(i) Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

(ii) The equation thus obtained will be of the form in which the variables are separable. Separate the variables and integrate.

(iii) After integration replace v by y/x and get the required solution.

Illustrative Examples

Example 7: *Solve* $(x^2 - y^2)\,dx + 2xy\,dy = 0.$ **(Gorakhpur 2008; Kumaun 13)**

Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \qquad \ldots(1)$$

This is a homogeneous differential equation as each term in the N^r and D^r of R.H.S. is of the second degree in x and y.

$\therefore$ putting $y = vx$ and consequently $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1), we get

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2vx^2} = \frac{v^2 - 1}{2v}$$

or $$x\frac{dv}{dx}=\frac{v^2-1}{2v}-v=\frac{v^2-1-2v^2}{2v}=\frac{-1-v^2}{2v}$$

or $$\frac{2v\,dv}{v^2+1}=-\frac{dx}{x},$$ in which the variables have been separated.

$\therefore$ integrating, we get

$$\log(v^2+1)=-\log x+\log c$$

or $$\log\{(v^2+1)\cdot x\}=\log c \quad \text{or} \quad x(v^2+1)=c$$

or $$x\{(y/x)^2+1\}=c \qquad [\text{putting } v=y/x]$$

or $(y^2+x^2)=cx$, which is the required solution.

Example 8: *Solve* $y^2\,dx+(xy+x^2)\,dy=0.$ **(Avadh 2010)**

Solution: The given equation can be written as

$$\frac{dy}{dx}=-\frac{y^2}{xy+x^2},$$ which is homogeneous.

$\therefore$ putting $y=vx$, this equation reduces to

$$\left(v+x\frac{dv}{dx}\right)=-\frac{v^2x^2}{x\,.\,vx+x^2}=-\frac{v^2}{v+1}$$

or $$x\frac{dv}{dx}=-\frac{v^2}{v+1}-v=-\frac{v^2+v^2+v}{v+1}=-\frac{2v^2+v}{v+1},$$

in which the variables are separable.

$\therefore$ $$-\frac{dx}{x}=\frac{(v+1)\,dv}{(2v^2+v)}=\frac{(v+1)}{v(2v+1)}dv.$$

Integrating, we get

$$-\log x+\log c=\int\frac{v+1}{v(2v+1)}dv,$$ where c is an arbitrary constant.

Now let $\dfrac{v+1}{v(2v+1)}=\dfrac{A}{v}+\dfrac{B}{2v+1}$. Then $A=1, B=-1.$

$\therefore$ $$\int\frac{v+1}{v(2v+1)}dv=\int\left[\frac{1}{v}-\frac{1}{2v+1}\right]dv=\log v-\frac{1}{2}\log(2v+1).$$

Hence the solution is

$$-\log x+\log c=\log v-\frac{1}{2}\log(2v+1)$$

or $$\frac{1}{2}\log(2v+1)=\log v+\log x-\log c$$

or $$\log(2v+1)^{1/2}=\log(vx/c)$$

or $$\{2(y/x)+1\}^{1/2}=y/c \qquad [\because v=y/x]$$

or $$(2y+x)/x=y^2/c^2 \quad \text{or} \quad c^2(2y+x)=xy^2.$$

Example 9: *Solve* $x^2 y\, dx - (x^3 + y^3)\, dy = 0.$

(Gorakhpur 2007; Garhwal 07; Agra 08; Kumaun 08; Kanpur 15)

Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}. \qquad \ldots(1)$$

This is a homogeneous equation as each term in the N^r and D^r on the R.H.S. is of the same degree (3rd degree).

$\therefore$ putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1), we get

$$v + x\frac{dv}{dx} = \frac{x^2 \,.\, vx}{x^3 + v^3 x^3} = \frac{v}{1 + v^3} \quad \text{or} \quad x\frac{dv}{dx} = \frac{v}{1+v^3} - v = -\frac{v^4}{1+v^3}$$

or $$\frac{dx}{x} = -\frac{(1+v^3)\, dv}{v^4} \quad \text{or} \quad \frac{dx}{x} = -\frac{1}{v^4}\, dv - \frac{1}{v}\, dv.$$

$\therefore$ integrating, we get

$$\log x + \log c = 1/(3v^3) - \log v$$

or $$\log (cxv) = 1/(3v^3)$$

or $$3 \log (cxv) = 1/v^3 \quad \text{or} \quad \log (cxv)^3 = 1/v^3$$

or $$c^3 x^3 v^3 = e^{1/v^3} \quad \text{or} \quad c^3 y^3 = e^{x^3/y^3} \qquad [\because v = y/x]$$

or $y^3 = ke^{x^3/y^3}$, which is the required solution.

Example 10: *Solve* $(x^3 - 3xy^2)\, dx = (y^3 - 3x^2 y)\, dy.$

(Meerut 2007; Kumaun 09; Garhwal 13)

Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2 y}, \qquad \ldots(1)$$

which is homogeneous.

$\therefore$ putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1), we get

$$v + x\frac{dv}{dx} = \frac{x^3 - 3x \,.\, v^2 x^2}{v^3 x^3 - 3x^2 \,.\, vx} = \frac{1 - 3v^2}{v^3 - 3v}$$

or $$x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v = \frac{1 - 3v^2 - v^4 + 3v^2}{v^3 - 3v} = \frac{1 - v^4}{v^3 - 3v}$$

or $$\frac{dx}{x} = \frac{v^3 - 3v}{1 - v^4}\, dv = \left[\frac{1}{2(v+1)} + \frac{1}{2(v-1)} - \frac{2v}{v^2 + 1}\right] dv,$$

(by partial fractions)

$\therefore$ integrating, we get

$$\log x + \log c = \frac{1}{2}\log (v+1) + \frac{1}{2}\log (v-1) - \log (v^2 + 1)$$

or $$\log (cx) = \log [\{(v+1)^{1/2} (v-1)^{1/2}\}/(v^2+1)]$$

or $$cx\,(v^2+1) = (v^2-1)^{1/2} \quad \text{or} \quad c^2x^2\,(v^2+1)^2 = v^2-1$$

or $$c^2x^2\,\{(y^2/x^2)+1\}^2 = \{(y^2/x^2)-1\}, \qquad [\because v = y/x]$$

or $$c^2\,(y^2+x^2)^2 = (y^2-x^2),$$ which is the required solution.

Example 11: *Solve* $x\left(\cos\dfrac{y}{x}\right)(y\,dx + x\,dy) = y\left(\sin\dfrac{y}{x}\right)(x\,dy - y\,dx).$

Solution: The given equation can be written as

$$x\left(\cos\frac{y}{x}\right)\left(y + x\frac{dy}{dx}\right) = y\left(\sin\frac{y}{x}\right)\left(x\frac{dy}{dx} - y\right)$$

or $$x\frac{dy}{dx}\left(x\cos\frac{y}{x} - y\sin\frac{y}{x}\right) = -y^2\sin\frac{y}{x} - xy\cos\frac{y}{x}$$

or $$\frac{dy}{dx} = \frac{y\,\{y\sin(y/x) + x\cos(y/x)\}}{x\{y\sin(y/x) - x\cos(y/x)\}} \qquad \ldots(1)$$

Now putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in (1), we get

$$v + x\frac{dv}{dx} = \frac{vx\,(vx\sin v + x\cos v)}{x\,(vx\sin v - x\cos v)}$$

or $$v + x\frac{dv}{dx} = \frac{v\,(v\sin v + \cos v)}{(v\sin v - \cos v)}$$

or $$x\frac{dv}{dx} = \frac{v^2\sin v + v\cos v - v^2\sin v + v\cos v}{v\sin v - \cos v},$$ variables separable

or $$-\left(\frac{-v\sin v + \cos v}{v\cos v}\right)dv = \frac{2\,dx}{x},$$ separating the variables.

$\therefore$ integrating, we get

$$-\log(v\cos v) = 2\log x + \log c$$

or $$\log\{1/(v\cos v)\} = \log(cx^2) \quad \text{or} \quad cx^2 = 1/(v\cos v)$$

or $$cx^2\,v\cos v = 1 \quad \text{or} \quad cx^2\,.\,(y/x)\cos(y/x) = 1$$

or $$cxy\cos(y/x) = 1,$$ which is the required solution.

Comprehensive Exercise 3

Solve the following differential equations :

1. $x + y\,(dy/dx) = 2y.$ **(Gorakhpur 2005)**
2. $y - x\,(dy/dx) = x + y\,(dy/dx).$ **(Gorakhpur 2010; Garhwal 08)**
3. $(x^2 + y^2)\,dx - 2xy\,dy = 0.$ **(Avadh 2006; Bundelkhand 03)**

4. $(x^2 + y^2)(dy/dx) = xy.$ **(Kanpur 2002; Kumaun 12)**

5. $dy/dx = y^2/(xy - x^2).$ **(Kumaun 2015)**

6. $x^2\, dy + y(x + y)\, dx = 0.$

7. $x(x - y)\, dy = y(x + y)\, dx.$

8. $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = y.$

9. $x\, dy - y\, dx = \sqrt{(x^2 + y^2)}\, dx.$ **(Meerut 2008; Lucknow 10)**

10. $(1 + e^{x/y})\, dx + e^{x/y}\{1 - (x/y)\}\, dy = 0.$ **(Meerut 2006)**

11. $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}.$

12. $x\dfrac{dy}{dx} = y - x\cos^2\dfrac{y}{x}.$

13. $x\left(\dfrac{dy}{dx}\right) = y + x\, e^{y/x}.$

14. $x\dfrac{dy}{dx}\cos\left(\dfrac{y}{x}\right) = y\cos\left(\dfrac{y}{x}\right) - x.$

15. $x\dfrac{dy}{dx} - y = x\sqrt{(x^2 + y^2)}.$ **(Garhwal 2010)**

16. $\dfrac{dy}{dx} + \dfrac{x^2 + 3y^2}{3x^2 + y^2} = 0.$ **(Garhwal 2009)**

Answers 3

1. $\log(y - x) = c + x/(y - x)$
2. $\frac{1}{2}\log(x^2 + y^2) + \tan^{-1}(y/x) = \log c$
3. $x^2 - y^2 = cx$
4. $cy = e^{x^2/(2y^2)}$
5. $y = ke^{y/x}$
6. $(y + 2x) = c^2x^2y$
7. $c^2xy = e^{-x/y}$
8. $cx = e^{x/y}$
9. $y + \sqrt{(y^2 + x^2)} = cx^2$
10. $x + ye^{x/y} = c$
11. $\sin(y/x) = cx$
12. $\tan(y/x) = \log(c/x)$
13. $e^{-y/x} + \log x = c$
14. $\sin\left(\dfrac{y}{x}\right) + \log x = c$
15. $y = x\sinh(x + c)$
16. $\log(x + y) + 2xy(x + y)^{-2} = c$

2.3 Equations Reducible To Homogeneous Form

A differential equation of the form

$$\frac{dy}{dx}=\frac{ax+by+c}{a_1x+b_1y+c_1}\text{, where } \frac{a}{a_1}\neq\frac{b}{b_1}$$

can be reduced to homogeneous form by taking new variables X and Y such that $x = X + h$ and $y = Y + k$, where h and k are constants to be so chosen as to make the given equation homogeneous. With the above substitutions we get $dx = dX$ and $dy = dY$, so that $dy/dx = dY/dX$. Hence the given equation becomes

$$\frac{dY}{dX}=\frac{a\,(X+h)+b\,(Y+k)+c}{a_1\,(X+h)+b_1\,(Y+k)+c_1}=\frac{aX+bY+(ah+bk+c)}{a_1X+b_1Y+(a_1h+b_1k+c_1)}.$$

Now choose h and k such that

$$ah+bk+c=0,\qquad\text{and}\qquad a_1h+b_1k+c_1=0.$$

Then the differential equation becomes

$$\frac{dY}{dX}=\frac{aX+bY}{a_1X+b_1Y}\text{, which is homogeneous.}$$

Now this equation can be solved as in article 2.2 substituting $Y = vX$. Finally by replacing X by $(x-h)$ and Y by $(y-k)$ we shall get the solution in original variables x and y.

If, however, $a/a_1 = b/b_1 = m$ (say) then the differential equation becomes of the form

$$\frac{dy}{dx}=\frac{m\,(a_1x+b_1y)+c}{a_1x+b_1y+c_1}.$$

To solve such a differential equation put $v = a_1x + b_1y$, get rid of y and then the transformed equation will be such that the variables are separable.

Illustrative Examples

Example 12: *Solve* $\dfrac{dy}{dx}=\dfrac{y-x+1}{y+x+5}$.

(Kumaun 2011)

Solution: Here $\dfrac{a}{a_1}\neq\dfrac{b}{b_1}$ *i.e.*, the coefficients of x and y in the numerator and denominator of the expression for (dy/dx) are not proportional. Such equations can be reduced to homogeneous form by taking new variables X and Y such that $x = X + h$ and $y = Y + k$. where h and k are constants to be taken at our choice. With these substitutions the given equation reduces to

$$\frac{dY}{dX}=\frac{Y+k-X-h+1}{Y+k+X+h+5}=\frac{Y-X+(k-h+1)}{Y+X+(k+h+5)}. \qquad \ldots(1)$$

Choose h and k such that

$$k-h+1=0\qquad\text{and}\qquad k+h+5=0.$$

Solving these, we get $k = -3, h = -2$.

With these values of h and k the equation (1) becomes

$$\frac{dY}{dX} = \frac{Y - X}{Y + X}. \qquad \ldots(2)$$

Now (2) is a homogeneous equation, so putting $Y = vX$, we have

$$v + X\frac{dv}{dX} = \frac{v-1}{v+1}, \quad \text{or} \quad X\frac{dv}{dX} = \frac{v-1}{v+1} - v = \frac{-1-v^2}{v+1}$$

or $$\frac{v+1}{v^2+1}\,dv = -\frac{dX}{X} \quad \text{or} \quad \frac{v}{v^2+1}\,dv + \frac{dv}{v^2+1} = -\frac{dX}{X}$$

or $$\frac{2v}{v^2+1}\,dv + \frac{2}{v^2+1}\,dv = -\frac{2}{X}\,dX.$$

Integrating, we have

$$\log(v^2+1) + 2\tan^{-1} v = -2\log X + c$$

or $$\log\{(v^2+1)X^2\} = -2\tan^{-1} v + c$$

or $$\log(Y^2 + X^2) = -2\tan^{-1}(Y/X) + c, \qquad [\because\ v = Y/X]$$

or $$\log[(y+3)^2 + (x+2)^2] + 2\tan^{-1}\{(y+3)/(x+2)\} = c,$$

$$[\because\ Y = y - k = y + 3, X = x - h = x + 2]$$

which is the required solution.

Example 13: *Solve* $\dfrac{dy}{dx} = \dfrac{2x+2y-2}{3x+y-5}$.

(Lucknow 2005)

Solution: Here $(a/a_1) \neq (b/b_1)$, therefore putting $x = X + h$ and $y = Y + k$ the given equation reduces to

$$\frac{dY}{dX} = \frac{2(X+h) + 2(Y+k) - 2}{3(X+h) + (Y+k) - 5} = \frac{2X + 2Y + (2h + 2k - 2)}{3X + Y + (3h + k - 5)}. \qquad \ldots(1)$$

Now choose h and k such that

$$2h + 2k - 2 = 0 \quad \text{and} \quad 3h + k - 5 = 0.$$

Solving these, we get $h = 2$ and $k = -1$.

With these values of h and k, the equation (1) becomes

$$\frac{dY}{dX} = \frac{2X + 2Y}{3X + Y}.$$

This is a homogeneous equation so putting $Y = vX$, we get

$$v + X\frac{dv}{dX} = \frac{2X + 2vX}{3X + vX} = \frac{2+2v}{3+v}$$

or $$X\frac{dv}{dX} = \frac{2+2v}{3+v} - v = \frac{2 + 2v - 3v - v^2}{3+v} = \frac{2 - v - v^2}{3+v}$$

or $$-\frac{dX}{X} = \frac{3+v}{v^2+v-2}\,dv = \frac{3+v}{(v+2)(v-1)}\,dv$$

$$= \left[\frac{4}{3(v-1)} - \frac{1}{3(v+2)}\right] dv \text{, by resolving into partial fractions}$$

or $$3\frac{dX}{X} = \left[\frac{1}{v+2} - \frac{4}{v-1}\right] dv.$$

Integrating, we get

$$3 \log X = \log(v+2) - 4\log(v-1) + \log c$$

or $$\log\{X^3 (v-1)^4\} = \log\{c(v+2)\}$$

or $$X^3 [(Y/X) - 1]^4 = c\{(Y/X) + 2\}, \qquad [\because v = Y/X]$$

or $$(Y - X)^4 = c(Y + 2X)$$

or $$\{(y+1) - (x-2)\}^4 = c\{(y+1) + 2(x-2)\}$$

or $(y - x + 3)^4 = c(2x + y - 3)$, which is the required solution.

Example 14: *Solve* $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0.$

(Lucknow 2011; Kanpur 12, 15)

Solution: We have $\dfrac{dy}{dx} = -\dfrac{2x+y+1}{4x+2y-1}$.

Here $\dfrac{a}{a_1} = \dfrac{b}{b_1}$.

$\therefore$ let $2x + y = v$ so that $2 + (dy/dx) = dv/dx$.

With these substitutions the given equation reduces to

$$\frac{dv}{dx} - 2 = -\frac{v+1}{2v-1},$$

or $$\frac{dv}{dx} = 2 - \frac{v+1}{2v-1} = \frac{3v-3}{2v-1}$$

or $$3\,dx = \frac{2v-1}{v-1}\,dv = \frac{2(v-1)+1}{v-1}\,dv = \left(2 + \frac{1}{v-1}\right) dv.$$

$\therefore$ integrating, we have

$$3x + c = 2v + \log(v-1)$$

or $$3x + c = 2(2x + y) + \log(2x + y - 1), \qquad [\because v = 2x + y]$$

or $x + 2y + \log(2x + y - 1) = c$, which is the required solution.

Comprehensive Exercise 4

Solve the following differential equations :

1. $(2x + y - 3)\,dy = (x + 2y - 3)\,dx$. **(Bundelkhand 2010; Lucknow 09; Rohilkhand 11)**
2. $dy/dx = (2x - y + 1)/(x + 2y - 3)$.
3. $(2x + 3y - 5)(dy/dx) + (3x + 2y - 5) = 0$.
4. $(x - y - 2)\,dx + (x - 2y - 3)\,dy = 0$.
5. $(x + y - 1)\,dy = (x + y)\,dx$.

6. $(x+y)(dx-dy)=dx+dy.$ **(Gorakhpur 2008)**

7. $(x-y-2)\,dx=(2x-2y-3)\,dy.$ **(Kashi 2011)**

8. $\dfrac{dy}{dx}=\dfrac{x-y+3}{2x-2y+5}.$ **(Garhwal 2002, 10)**

9. $(2x+4y+1)\,dy=(2y+x+1)\,dx.$

10. $(2x+y+1)\,dx+(4x+2y-1)\,dy=0.$ **(Lucknow 2011)**

Answers 4

1. $(x+y-2)=c^2\,(x-y)^3$
2. $(5y-7)^2+(5x-1)(5y-7)-(5x-1)^2=c$
3. $3x^2+4xy+3y^2-10x-10y=c$
4. $\log\{c^2\,(2Y^2-X^2)\}=(1/\sqrt{2})\log\{(Y\sqrt{2}-X)/(Y\sqrt{2}+X)\},$ where $X=x-1$ and $Y=y+1.$
5. $2(y-x)-\log(2x+2y-1)=c$
6. $(y-x)+\log(x+y)=c$
7. $\log(x-y-1)=x-2y-c$
8. $x-2y+\log(x-y+2)=c$
9. $4(2y-x)-\log(4x+8y+3)=c$
10. $x+2y+\log(2x+y-1)=c$

2.4. Linear Differential Equations

Definition: *A differential equation is said to be linear when the dependent variable y and all its derivatives occur in the first degree only and are not multiplied together.*

An equation of the form

$$\frac{dy}{dx}+Py=Q,$$

where P and Q are functions of x only is called a linear differential equation of the first order with y as the dependent variable. To solve such an equation, multiply both the sides by $e^{\int P\,dx}$. The equation then becomes

$$e^{\int P\,dx}(dy/dx)+Pye^{\int P\,dx}=Qe^{\int P\,dx}$$

or

$$\frac{d}{dx}\left\{ye^{\int P\,dx}\right\}=Qe^{\int P\,dx}.$$

Integrating both sides w.r.t., 'x', we get

$$ye^{\int P\,dx}=\int Qe^{\int P\,dx}dx+c,$$

which is the solution of the differential equation.

Note 1: The factor $e^{\int P\,dx}$, on multiplying by which the L.H.S. of the differential equation becomes a differential coefficient of some function of x and y, is called an **integrating factor** (I.F.) of the differential equation. **(Agra 2007)**

Note 2: Sometimes a given differential equation becomes *linear if we take y as the independent variable and x as the dependent variable.*

Then it is of the form $\dfrac{dx}{dy} + Px = Q$,

where P and Q are some functions of y only. The integrating factor (I.F.) in this case is $e^{\int P\,dy}$ and the solution is

$$x \cdot e^{\int P\,dy} = \int Q \cdot e^{\int P\,dy} dy + c.$$

Working Rule :

(i) Write the given equation in the form

$$\frac{dy}{dx} + Py = Q \qquad \textbf{or} \qquad \frac{dx}{dy} + Px = Q \text{ as the case may be.}$$

(ii) Find the integrating factor $e^{\int P\,dx}$ **or** $e^{\int P\,dy}$.

(iii) The solution of the differential equation is either

$$y \cdot (\text{I. F.}) = \int \{Q \cdot (\text{I. F.})\}\, dx + c$$

or $$x \cdot (\text{I. F.}) = \int \{Q \cdot (\text{I. F.})\}\, dy + c \text{ as the case may be.}$$

Illustrative Examples

Example 15: *Solve* $\dfrac{dy}{dx} + 2xy = e^{-x^2}$.

(Meerut 2001, 03, 10B; Kumaun 06)

Solution: The given differential equation is $\dfrac{dy}{dx} + 2xy = e^{-x^2}$, which is linear with y as the dependent variable.

Here $P = 2x$, and $Q = e^{-x^2}$.

We have $\int P\,dx = \int 2x\,dx = 2 \cdot \frac{1}{2} x^2 = x^2$.

Therefore I.F. $= e^{\int P\,dx} = e^{x^2}$.

Hence the solution is $y.\,(\text{I.F.}) = \int \{Q \cdot (\text{I. F.})\}\, dx + c$, where c is an arbitrary constant

or $$y \cdot e^{x^2} = \int (e^{-x^2} . e^{x^2})\, dx + c$$

or $$y \cdot e^{x^2} = \int dx + c \qquad \text{or} \qquad ye^{x^2} = x + c.$$

Example 16: *Solve* $(x^2 - 1)(dy/dx) + 2xy = 1.$ **(Meerut 2010)**

Solution: The given equation can be written as

$$\frac{dy}{dx}+\frac{2x}{x^2-1}\,y=\frac{1}{x^2-1} \qquad \left[\text{From } \frac{dy}{dx}+Py=Q\right]$$

Here $P=\dfrac{2x}{x^2-1}$, and so $\int P\,dx=\int \dfrac{2x\,dx}{x^2-1}=\log(x^2-1)$.

$\therefore$ integrating factor (I.F.) $= e^{\int P\,dx}=e^{\log(x^2-1)}=(x^2-1)$.

$\therefore$ the solution is $y.(x^2-1)=\int \dfrac{1}{(x^2-1)}\cdot(x^2-1)\,dx+c,$ $\qquad \left[\because\ Q=\dfrac{1}{(x^2-1)}\right]$

or $\qquad y(x^2-1)=\int dx+c=x+c.$

Hence $y(x^2-1)=x+c$ is the required solution.

Example 17: *Solve* $x(dy/dx)+2y=x^2\log x$.

Solution: The given equation can be written as

$$\frac{dy}{dx}+\frac{2}{x}\,y=x\log x, \qquad \left(\text{from } \frac{dy}{dx}+Py=Q\right)$$

Here $P=(2/x)$ and so $\int P\,dx=\int(2/x)\,dx=2\log x=\log x^2$.

$\therefore$ $\qquad$ I.F. $=e^{\int P\,dx}=e^{\log x^2}=x^2$.

$\therefore$ the solution is

$$y\,(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dx+c$$

i.e., $\qquad yx^2=\int(x\log x)\,x^2\,dx+c,$ $\qquad [\because\ Q=x\log x]$

or $\qquad yx^2=\int x^3\log x\,dx+c$

or $\qquad yx^2=(\log x)\cdot\dfrac{x^4}{4}-\int\dfrac{1}{x}\cdot\dfrac{x^4}{4}\,dx+c,$ $\qquad$ [integrating by parts]

or $\qquad yx^2=\dfrac{1}{4}x^4\log x-\dfrac{1}{4}\cdot(\dfrac{1}{4}x^4)+c$

Hence $4yx^2=x^4\log x-\dfrac{1}{4}x^4+k$ is the required solution.

Example 18: *Solve* $\cos^2 x\,(dy/dx)+y=\tan x$.

(Meerut 2003, 13; Avadh 05; Garhwal 06)

Solution: The given equation can be written as $\dfrac{dy}{dx}+\sec^2 x\,.\,y=\tan x\sec^2 x$, which is linear with y as the dependent variable.

Here $P=\sec^2 x$, and $Q=\tan x\sec^2 x$.

We have $\qquad \int P\,dx=\int\sec^2 x\,dx=\tan x.$

$\therefore$ I. F. $= e^{\int P\,dx} = e^{\tan x}$.

$\therefore$ the solution is $y\,.\,e^{\tan x} = \int \tan x \sec^2 x\,.\,e^{\tan x}\,dx + c$

$$\text{or}\qquad ye^{\tan x} = \int t\,e^t dt + c, \text{ putting } \tan x = t \text{ and } \sec^2 x\,dx = dt$$

$$= [t\,e^t - \int e^t\,dt] + c, \text{ integrating by parts}$$

$$= t\,e^t - e^t + c = \tan x\,e^{\tan x} - e^{\tan x} + c$$

$$\text{or}\qquad ye^{\tan x} = e^{\tan x}(\tan x - 1) + c.$$

Example 19: *Solve* $(1 + y^2)\,dx = (tan^{-1}\,y - x)\,dy$. **(Agra 2005; Avadh 07; Rohilkhand 10; Bundelkhand 03, 04; Meerut 13B; Garhwal 11)**

Solution: The given equation can be written as

$$\frac{dx}{dy} = \frac{\tan^{-1} y - x}{1 + y^2} \qquad \text{or} \qquad \frac{dx}{dy} + \frac{1}{1 + y^2}\cdot x = \frac{\tan^{-1} y}{1 + y^2}.$$

This is linear equation with y as independent variable and x as dependent variable, Here $P = 1/(1 + y^2)$.

$$\therefore \qquad \int P\,dy = \int \{1/(1 + y^2)\}\,dy = \tan^{-1} y.$$

Hence I.F. $= e^{\int P\,dy} = e^{\tan^{-1} y}$.

Now the solution is $x\,.\,(\text{I. F.}) = \int \{Q\,.\,(\text{I. F.})\}\,dy + c$

$$i.e., \qquad x\,e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1 + y^2}\cdot e^{\tan^{-1} y} dy + c, \qquad \left(\because\ Q = \frac{\tan^{-1} y}{1 + y^2}\right)$$

$$= \int t\,e^t\,dt + c, \text{ putting } \tan^{-1} y = t \text{ and } dy/(1 + y^2) = dt$$

$$= [t\,e^t - \int 1\,.\,e^t\,dt] + c, \text{ integrating by parts}$$

$$= t\,e^t - e^t + c = (t - 1)\,e^t + c$$

$$\text{or}\qquad x\,.\,e^{\tan^{-1} y} = (\tan^{-1} y - 1)\,.\,e^{\tan^{-1} y} + c.$$

Example 20: *Solve* $(dy/dx) + 2\,y \tan x = \sin x$, *given that* $y = 0$ *when* $x = \pi/3$.

Solution: We have $\dfrac{dy}{dx} + 2 \tan x\,.\,y = \sin x$, which is linear with y as dependent variable.

Here $P = 2 \tan x$ and so $\int P\,dx = \int 2 \tan x\,dx = 2 \log \sec x = \log \sec^2 x$.

$$\therefore \qquad \text{I.F.} = e^{\int P\,dx} = e^{\log \sec^2 x} = \sec^2 x.$$

$\therefore$ the solution is $y\,.\,(\text{I.F.}) = \int [Q\,.\,(\text{I.F.})]\,dx + c$

i.e., $$y \sec^2 x = \int \sin x\,.\sec^2 x\,dx + c = \int \tan x \sec x\,dx + c$$

or $$y \sec^2 x = \sec x + c. \qquad \ldots(1)$$

Now it is given that $y = 0$ at $x = \frac{1}{3}\pi$.

Therefore, from (1)

$$0\,.\sec^2 \frac{1}{3}\pi = \sec \frac{1}{3}\pi + c \qquad \text{or} \qquad c = -\sec \frac{1}{3}\pi = -2.$$

Hence from (1), $y \sec^2 x = \sec x - 2$ is the required solution.

Comprehensive Exercise 5

Solve the following differential equations :

1. $(dy/dx) + (1/x)\,y = x^n$.

2. $x^2\,(dy/dx) + y = 1$.

3. $x\,(dy/dx) + y = x^2 + 3x + 2$ **(Avadh 2010)**

4. $(\sin x)\,(dy/dx) + 3y = \cos x$. **(Meerut 2004)**

5. $(x^2 + 1)\,(dy/dx) + 2xy = 4x^2$. **(Agra 2008; Kumaun 10)**

6. $(1 + x^2)\,(dy/dx) + 2xy = \cos x$. **(Meerut 2009; Kumaun 09, 12)**

7. $x\,(x^2 + 1)\,(dy/dx) = y\,(1 - x^2) + x^3 \log x$. **(Meerut 2007B)**

8. $(dy/dx) + (2y/x) = \sin x$.

9. $(dy/dx) + (y/x) = \sin x^2$

10. $(y \sin x - 1)\,dx + \cos x\,dy = 0$. **(Bundelkhand 2008)**

11. $\sin 2x\,(dy/dx) - y = \tan x$. **(Agra 2007)**

12. $(1 + x)\,(dy/dx) - xy = 1 - x$.

13. $(x + 2y^3)\,dy = y\,dx$. **(Rohilkhand 2006; Avadh 11)**

14. $(x + y + 1)\,(dy/dx) = 1$. **(Gorakhpur 2006; Rohilkhand 08)**

15. $(1 + y^2)\,dx + (x - e^{\tan^{-1} y})\,dy = 0$. **(Gorakhpur 2007; Lucknow 09; Purvanchal 11; Bundelkhand 04)**

Answers 5

1. $xy = x^{n+2}/(n+2) + c$

2. $y = 1 + ce^{1/x}$

3. $6xy = 2x^3 + 9x^2 + 12x + c$

4. $y \tan^3 (\frac{1}{2} x) = 2 \tan (\frac{1}{2} x) - x - \frac{1}{3} \tan^3 (\frac{1}{2} x) + c$

5. $y (1 + x^2) = \frac{4}{3} x^3 + c$

6. $y (1 + x^2) = \sin x + c$

7. $y (x^2 + 1) = cx + \frac{1}{2} x^3 \log x - \frac{1}{4} x^3$

8. $yx^2 = - x^2 \cos x + 2x \sin x + 2 \cos x + c$

9. $yx = - \frac{1}{2} \cos x^2 + c$

10. $y \sec x = \tan x + c$

11. $y = \tan x + c \sqrt{} (\tan x)$

12. $y (1 + x) = x + ce^x$

13. $x = y^3 + cy$

14. $x = ce^y - y - 2$

15. $xe^{\tan^{-1} y} = \frac{1}{2} e^{2 \tan^{-1} y} + c$

2.5. Equations Reducible to the Linear Form

Some equations can be reduced to the linear form by making suitable substitutions, and hence can be solved easily.

(a) Bernoulli's Equation: **(Gorakhpur 2008)**

A differential equation of the form

$$\frac{dy}{dx} + Py = Q\, y^n,$$

where P and Q are functions of x alone is called Bernoulli's equation.

To solve this equation dividing both sides by y^n, we get

$$y^{-n} \frac{dy}{dx} + Py^{-n+1} = Q$$

[Note that the R.H.S. is now a function of x alone.]

Now putting $y^{-n+1} = v$, so that $(1 - n)\, y^{-n} \frac{dy}{dx} = \frac{dv}{dx}$, the equation (1) transforms to

$$\frac{1}{1-n} \frac{dv}{dx} + Pv = Q$$

or
$$\frac{dv}{dx} + (1 - n)\, P \,.\, v = (1 - n)\, Q,$$

which is a linear differential equation with v as dependent variable and can be solved by the method discussed in article 2.4.

(b) Equations of the form

$$\frac{dy}{dx} + P\phi\, (y) = Q\, f\, (y),$$

where P and Q are functions of x alone.

To reduce such an equation to the linear form, dividing both sides by $f\, (y)$ and thus getting rid of $f\, (y)$ from the R.H.S., we have

$$\frac{1}{f(y)} \cdot \frac{dy}{dx} + P \cdot \frac{\phi(y)}{f(y)} = Q.$$

Now we try the substitution $\dfrac{\phi(y)}{f(y)} = v \cdot$

If $$\frac{dv}{dx} = \frac{d}{dx}\left\{\frac{\phi(y)}{f(y)}\right\} = K \cdot \frac{1}{f(y)} \frac{dy}{dx},$$

where K is some constant, the equation reduces to the form $\dfrac{1}{K}\dfrac{dv}{dx} + Pv = Q$, or $\dfrac{dv}{dx} + K\,Pv = KQ$, which is a linear differential equation, with v as the dependent variable.

Illustrative Examples

Example 21: *Solve* $\dfrac{dy}{dx} + \dfrac{1}{x}\,y = x^2 y^6.$

Solution: The given equation on dividing out by y^6, becomes

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{y^5} = x^2. \qquad \ldots(1)$$

Put $\dfrac{1}{y^5} = v$, so that

$$-\frac{5}{y^6}\frac{dy}{dx} = \frac{dv}{dx} \quad \text{or} \quad \frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5}\frac{dv}{dx}.$$

With these substitutions the equation (1) becomes

$$-\frac{1}{5}\frac{dv}{dx} + \frac{1}{x} \cdot v = x^2 \quad \text{or} \quad \frac{dv}{dx} - \frac{5}{x}\,v = -5x^2.$$

This is a linear equation with v as the dependent variable. Here $P = -5/x$ and $Q = -5x^2$.

We have

$$\int P\,dx = \int (-5/x)\,dx = -5 \log x = \log x^{-5} = \log(1/x^5).$$

$\therefore$ I.F. $= e^{\int P\,dx} = e^{\log(1/x^5)} = 1/x^5.$

$\therefore$ the solution is

$$v \cdot (\text{I.F.}) = \int \{Q \cdot (\text{I.F.})\}\,dx + c$$

i.e., $$v \cdot (1/x^5) = \int -5x^2 \cdot (1/x^5)\,dx + c = -5\int x^{-3}\,dx + c$$

i.e., $$(1/y^5) \cdot (1/x^5) = \frac{5}{2}x^{-2} + c. \qquad [\text{Note that } v = 1/y^5]$$

Hence $1/(x^5 y^5) = \dfrac{5}{2} \cdot (1/x^2) + c$ is the required solution.

Example 22: *Solve* $\frac{dy}{dx}+\left(\frac{x}{1-x^2}\right)y = x\sqrt{y}$.

Solution: Dividing both sides of the given equation by $2y^{1/2}$, we have

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx}+\frac{1}{2}\left(\frac{x}{1-x^2}\right)\sqrt{y}=\frac{1}{2}x. \qquad \ldots(1)$$

Now put $y^{1/2}=v$, so that $\frac{1}{2}y^{-1/2}(dy/dx)=(dv/dx)$.

With these substitutions the equation (1) becomes

$$\frac{dv}{dx}+\frac{1}{2}\left(\frac{x}{1-x^2}\right)\cdot v=\frac{1}{2}x,$$

which is linear with v as the dependent variable.

Here $P=\frac{1}{2}\{x/(1-x^2)\}$ and $Q=\frac{1}{2}x$.

We have $\int P\,dx=\int\frac{1}{2}\frac{x}{1-x^2}dx=-\frac{1}{4}\int\frac{-2x}{1-x^2}dx$

$$=-\frac{1}{4}\log(1-x^2)=\log(1-x^2)^{-1/4}=\log\{1/(1-x^2)^{1/4}\}.$$

$\therefore$ I.F. $=e^{\int P\,dx}=e^{\log\{1/(1-x^2)^{1/4}\}}=1/(1-x^2)^{1/4}$.

$\therefore$ the solution is

$$v/(1-x^2)^{1/4}=\frac{1}{2}\int\{x/(1-x^2)^{1/4}\}\,dx+c$$

or $v/(1-x^2)^{1/4}=-\frac{1}{4}\int t^{-1/4}dt$, putting $(1-x^2)=t$ so that $-2x\,dx=dt$

or $\sqrt{y}/(1-x^2)^{1/4}=-\frac{1}{4}\cdot\frac{4}{3}t^{3/4}+c$, $[\because v=\sqrt{y}]$

or $\sqrt{y}/(1-x^2)^{1/4}=-\frac{1}{3}(1-x^2)^{3/4}+c$. $[\because t=(1-x^2)]$

Example 23: *Solve* $x(dy/dx)+y=y^2\log x$.

(Bundelkhand 2006; Gorakhpur 10; Purvanchal 07, 08, 10; Kanpur 06, 15)

Solution: The given equation can be written as

$$\frac{dy}{dx}+\frac{1}{x}y=\frac{y^2}{x}\log x,$$ dividing throughout by x

or
$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{x}\cdot\frac{1}{y}=\frac{\log x}{x}, \qquad \ldots(1)$$

dividing throughout by y^2.

Now put $1/y=v$, so that $(-1/y^2)(dy/dx)=(dv/dx)$.

With these substitutions equation (1) becomes

$$-\frac{dv}{dx}+\frac{1}{x}v=\frac{\log x}{x}$$

or $$\frac{dv}{dx} - \frac{1}{x} \cdot v = -\frac{\log x}{x}.$$

This is linear with v as the dependent variable.

Here $P = -1/x$ and $Q = -(\log x)/x$.

$\therefore$ I.F. $= e^{\int P\,dx} = e^{\int(-1/x)\,dx} = e^{-\log x} = e^{\log(1/x)} = (1/x)$.

Hence the solution is

$$v\,(1/x) = \int \{-(\log x)/x\} \cdot (1/x)\,dx + c$$

or $$v/x = -\int (1/x^2) \log x\,dx + c$$

$$= -\left[(-1/x)\log x - \int (1/x)(-1/x)\,dx\right] + c,$$

integrating by parts taking $1/x^2$ as the second function

$$= (1/x)\log x - \int (1/x^2)\,dx + c = (1/x)\log x + (1/x) + c$$

or $$1/(xy) = (1/x)(1 + \log x) + c, \qquad [\because v = 1/y]$$

or $$1 = y\,(1 + \log x) + cxy.$$

Example 24: *Solve* $\dfrac{dy}{dx} + \dfrac{y}{x} \log y = \dfrac{y}{x^2} (\log y)^2.$

(Kumaun 2011)

Solution: Dividing both sides of the given equation by $y\,(\log y)^2$, we have

$$\frac{1}{y\,(\log y)^2} \frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{\log y} = \frac{1}{x^2} \qquad \ldots(1)$$

Now put $1/\log y = v$, so that

$$-\{1/(\log y)^2\} \cdot (1/y) \cdot (dy/dx) = dv/dx.$$

With these substitutions the equation (1) becomes

$$-\frac{dv}{dx} + \frac{v}{x} = +\frac{1}{x^2} \qquad \text{or} \qquad \frac{dv}{dx} - \frac{1}{x} \cdot v = -\frac{1}{x^2}.$$

This is linear with v as the dependent variable.

Here $P = -1/x$ and $Q = -1/x^2$,

$\therefore$ I.F. $= e^{\int P\,dx} = e^{-\int(1/x)dx} = e^{-\log x} = e^{\log(1/x)} = 1/x$.

$\therefore$ the solution is $v/x = \int (-1/x^2) . (1/x) dx + c$

or $$(1/\log y)(1/x) = -\int x^{-3} dx + c, \qquad [\because v = 1/\log y]$$

or $$\frac{1}{x \log y} = \frac{1}{2x^2} + c.$$

Example 25: *Solve* $\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2}.$

Solution: On dividing out by e^y, the given equation becomes

$$e^{-y}\frac{dy}{dx}+e^{-y}\cdot\frac{1}{x}=\frac{1}{x^2}. \qquad \ldots(1)$$

Now put $e^{-y}=v$, so that $-e^{-y}(dy/dx)=(dv/dx)$.

With these substitutions equation (1) becomes

$$-\frac{dv}{dx}+\frac{1}{x}\cdot v=\frac{1}{x^2} \quad \text{or} \quad \frac{dv}{dx}-\frac{1}{x}\cdot v=-\frac{1}{x^2}.$$

This is linear with v as the dependent variable.

Here $P=-1/x$ and $Q=-1/x^2$.

$\therefore$ I.F. $=e^{\int P\,dx}=e^{\int -(1/x)\,dx}=e^{-\log x}=e^{\log(1/x)}=1/x$.

$\therefore$ the solution is $v.\,(\text{I.F.})=\int\{Q\,.\,(\text{I. F.})\}\,dx+c$

i.e., $v\,.\,\dfrac{1}{x}=-\displaystyle\int\frac{1}{x^2}\cdot\frac{1}{x}\,dx+c=\frac{1}{2x^2}+c$

or $e^{-y}\,.\,(1/x)=1/(2x^2)+c$ $\qquad[\because\ v=e^{-y}]$

or $e^{-y}(1/x)=(1+2cx^2)/(2x^2)$

or $2x=(2cx^2+1)\,e^y$.

Comprehensive Exercise 6

Solve the following differential equations :

1. $(dy/dx)+(y/x)=y^2$. **(Bundelkhand 2007; Garhwal 09, 11)**

2. $(dy/dx)+\{y/(x-1)\}=xy^{1/3}$. **(Meerut 2005B)**

3. $(x^3y^2+xy)\,dx=dy$. **(Kanpur 2002)**

4. $dy/dx=x^3y^3-xy$. **(Gorakhpur 2005; Meerut 13B)**

5. $3\,(dy/dx)+2\,y/(x+1)=x^3/y^2$.

6. $y\,(2xy+e^x)\,dx-e^x\,dy=0$. **(Kanpur 2001)**

7. $(xy^2-e^{1/x^3})\,dx-x^2y\,dy=0$. **(Lucknow 2007)**

8. $dy/dx=e^{x-y}(e^x-e^y)$.

9. $(dy/dx)-y\tan x=-y^2\sec x$. **(Meerut 2009B; Garhwal 10)**

10. $\cos x\,dy=(\sin x-y)\,y\,dx$. **(Rohilkhand 2007)**

11. $(dy/dx)+x\sin 2y=x^3\cos^2 y$. **(Gorakhpur 2007, 11; Purvanchal 08; Avadh 14)**

12. $(dy/dx)+y\cot x=y^2\sin^2 x$.

13. $2\,(dy/dx)-y\sec x=y^3\tan x$.

14. $(1-x^2)\dfrac{dy}{dx} + xy = xy^2.$ **(Purvanchal 2006)**

15. $x\dfrac{dy}{dx} + y \log y = xye^x.$ **(Bundelkhand 2005)**

16. Solve $xy^2\,(dy/dx) - 2y^3 = 2x^3$, given that $y = 1$ when $x = 1$.

17. Solve $\dfrac{dy}{dx} + y\dfrac{d\phi}{dx} = \phi(x)\dfrac{d\phi}{dx}$, where ϕ is some function of x only.

Answers 6

1. $xy\,(c - \log x) = 1$

2. $y^{2/3}(x-1)^{2/3} = \dfrac{2}{5}x\,(x-1)^{5/2} - \dfrac{3}{20}(x-1)^{8/3} + c$

3. $1 = 2y\,(1 - \frac{1}{2}x^2) - cy\,e^{-\frac{1}{2}x^2}$

4. $(1/y^2) = ce^{x^2} + x^2 + 1$

5. $y^3\,(x+1)^2 = \dfrac{1}{6}x^6 + \dfrac{2}{5}x^5 + \dfrac{1}{4}x^4 + c$

6. $e^x + y\,(c + x^2) = 0$

7. $3y^2 = 2x^2\,e^{1/x^3} + 3cx^2$

8. $e^y = e^x - 1 + ce^{-(e^x)}$

9. $\sec x = y\,(\tan x - c)$

10. $\sec x = y\,(\tan x - c)$

11. $2\tan y = (x^2 - 1) + 2\,ce^{-x^2}$

12. $(1/y) = (\cos x - c)\sin x$

13. $-(\sec x + \tan x) = y^2\,(\sec x + \tan x - x + c)$

14. $1/y^{n-1} = 2\sin x - \{2/(1-n)\} + ce^{(n-1)\sin x}$

15. $x \log y = e^x\,(x-1) + c$

16. $y^3 = -2x^3 + 3x^6$

17. $ye^{\phi} = \int \phi\, e^{\phi}\, d\phi + c$

2.6. Exact Differential Equations

(Lucknow 2006)

A differential equation is said to be exact if it can be obtained from its primitive (solution) directly by differentiation, without involving any subsequent process of multiplication, elimination, etc. Thus the differential equation of the form $M\,dx + N\,dy = 0$, where M and N are some functions of x and y, is **exact** if it can be obtained directly by differentiating an equation of the form $u = c$, where u is some function of x and y and c is an arbitrary constant.

Theorem: *The necessary and sufficient condition for the ordinary differential equation $M\,dx + N\,dy = 0$ to be exact is that* **(Lucknow 2006, 09)**

$$\partial M/\partial y = \partial N/\partial x\,.$$

The condition is necessary: Suppose the differential equation

$$M\,dx + N\,dy = 0 \qquad \ldots(1)$$

is exact. Let the primitive of (1) be $u = c$, where u is some function of x and y and c is an arbitrary constant.

Since u is a function of x and y, therefore from partial differentiation, we have

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy\,.$$

Now $$u = c \Rightarrow du = 0 \Rightarrow \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy = 0. \qquad \ldots(2)$$

Since the equation (1) is exact, therefore equation (2) must be identical with (1), therefore, we have

$$\frac{\partial u}{\partial x} = M \qquad \ldots(3)$$

and $$\frac{\partial u}{\partial y} = N. \qquad \ldots(4)$$

Differentiating (3) and (4) partially with respect to y and x respectively, we obtain

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial M}{\partial y}, \frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial N}{\partial x}\,.$$

But $$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x\, \partial y}\,.\text{ Therefore } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\,.$$

Hence the condition is necessary.

The condition is sufficient: We have to show that if $\partial M/\partial y = \partial N/\partial x$, then $M\, dx + N\, dy = 0$ must be an exact differential equation. Let $\int M\, dx = P$, then

$$\frac{\partial P}{\partial x} = M, \text{ so that } \frac{\partial^2 P}{\partial y\, \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\text{, by hypothesis.}$$

Now $$\frac{\partial N}{\partial x} = \frac{\partial^2 P}{\partial x\, \partial y} \Rightarrow \frac{\partial N}{\partial x} = \frac{\partial^2 P}{\partial x\, \partial y} \Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right)$$

$\Rightarrow$ $$N = \frac{\partial P}{\partial y} + f\,(y), \text{ where } f\,(y) \text{ is some function of } y \text{ alone.}$$

Now putting $M = \dfrac{\partial P}{\partial x}$ and $N = \dfrac{\partial P}{\partial y} + f\,(y)$ in $M\, dx + N\, dy$, we have

$$M\, dx + N\, dy = \frac{\partial P}{\partial x}\, dx + \left\{\frac{\partial P}{\partial y} + f\,(y)\right\} dy$$

$$= \frac{\partial P}{\partial x}\, dx + \frac{\partial P}{\partial y}\, dy + f\,(y)\, dy = dP + f\,(y)\, dy$$

$= d\,[P + F\,(y)]$, where $d\,[F\,(y)] = f\,(y)\, dy$

= an exact differential of some function of x and y.

Therefore the differential equation $M\, dx + N\, dy = 0$ is exact and hence the condition is sufficient.

Working rule for solving an exact differential equation: **(Meerut 2008)**

To solve a differential equation of the form $M\,dx + N\,dy = 0$, first ascertain with the help of the condition $\partial M/\partial y = \partial N/\partial x$ whether the equation is exact or not. If the equation is exact, then

(i) Integrate M with respect to x treating y as constant.

(ii) Integrate w.r.t. y only those terms of N which do not contain x.

(iii) Equate the sum of these two integrals [found in (i) and (ii)] to an arbitrary constant and thus get the required solution.

Thus if the differential equation $M\,dx + N\,dy = 0$ is exact, its solution is

$$\underset{\text{treating } y \text{ as constant}}{\int M\,dx} + \underset{\substack{\text{take only those terms in} \\ N \text{ which do not contain } x.}}{\int N\,dy} = c$$

Illustrative Examples

Example 26: *Solve* $(ax + hy + g)\,dx + (hx + by + f)\,dy = 0.$ **(Meerut 2003)**

Solution: Here $M = ax + hy + g$ and $N = hx + by + f$.

We have $\dfrac{\partial M}{\partial y} = h$ and $\dfrac{\partial N}{\partial x} = h$. Thus we observe that $\partial M/\partial y = \partial N/\partial x$ and therefore the given differential equation is exact. Hence its solution is

$$\underset{\text{treating } y \text{ as a constant}}{\int M\,dx} + \underset{\substack{\text{taking only those terms in} \\ N \text{ which do not contain } x.}}{\int N\,dy} = c$$

i.e., $$\underset{\text{treating } y \text{ as a constant}}{\int (ax + hy + g)\,dx} + \int (by + f)\,dy = c$$

i.e., $$\frac{1}{2}ax^2 + hxy + gx + \frac{1}{2}by^2 + fy = c$$

i.e., $$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\,,$$

where c is an arbitrary constant. We have replaced $-2c$ by c.

Example 27: *Solve* $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$.

(Purvanchal 2009; Meerut 11)

Solution: Here $M = 4x + 3y + 1$ and $N = 3x + 2y + 1$.

We have $\dfrac{\partial M}{\partial y} = 3$ and $\dfrac{\partial N}{\partial x} = 3$ *i.e.*, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, and hence the given equation is exact.

Integrating M, *i.e.*, $4x+3y+1$ with respect to x, treating y as a constant, we obtain $4\cdot\frac{1}{2}x^2+3xy+x$ *i.e.*, $2x^2+3xy+x$.

Again, the only terms in N which do not contain x are $2y+1$.

Integrating $(2y+1)$ with respect to y, we obtain

$$2\cdot\frac{1}{2}y^2+y \text{ i.e., } y^2+y.$$

Hence the solution of the given differential equation is

$$2x^2+3xy+x+y^2+y=c, \text{ where } c \text{ is an arbitrary constant}$$

i.e.,
$$2x^2+3xy+y^2+x+y=c.$$

Example 28: *Solve* $(x^2-ay)\,dx-(ax-y^2)\,dy=0$.

Solution: Here $M=x^2-ay$ and $N=-(ax-y^2)$.

We have

$$\frac{\partial M}{\partial y}=-a \text{ and } \frac{\partial N}{\partial x}=-a \text{ i.e., } \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

and hence the given equation is exact.

Now
$$\int M\,dx \text{ (regarding } y \text{ as a constant)} = \int (x^2-ay)\,dx=\frac{1}{3}x^3-ayx \qquad \ldots(1)$$

and
$$\int N\,dy \text{ (taking in } N \text{ only those terms which do not contain } x\text{)}$$
$$=\int y^2\,dy=\frac{1}{3}y^3\text{s} \qquad \ldots(2)$$

∴ the required solution is,

$$(1)+(2)=c \text{ (an arbitrary constant)}$$

or
$$\frac{1}{3}x^3-ayx+\frac{1}{3}y^3=c,$$

or
$$x^3+y^3-3axy=3c.$$

Example 29: *Solve* $x\,dx+y\,dy+\dfrac{x\,dy-y\,dx}{x^2+y^2}=0$.

(Bundelkhand 2007)

Solution: The given equation can be written as

$$\left[x-\frac{y}{x^2+y^2}\right]dx+\left[y+\frac{x}{x^2+y^2}\right]dy=0.$$

Here

$$M=x-\frac{y}{x^2+y^2} \text{ and } N=y+\frac{x}{x^2+y^2}.$$

We have

$$\frac{\partial M}{\partial y} = 0 - \frac{1 \cdot (x^2 + y^2) - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

and $$\frac{\partial N}{\partial x} = 0 + \frac{1 \cdot (x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Thus $\partial M / \partial y = \partial N / \partial x$ and hence the given equation is exact. Therefore its solution is

$$\int \left[x - \frac{y}{x^2 + y^2} \right] dx \qquad + \qquad \int y \ dy \qquad = c$$

regarding y as a constant

taking only those terms in N which do not contain x

or $$\frac{x^2}{2} - y \cdot \frac{1}{y} \tan^{-1} \frac{x}{y} + \frac{y^2}{2} = c$$

or $$x^2 - 2 \tan^{-1} (x/y) + y^2 = 2c = k.$$

Comprehensive Exercise 7

Solve the following differential equations :

1. $dy/dx = (2x - y)/(x + 2y - 5)$.
2. $(2ax + by) \ y \ dx + (ax + 2by) \ x \ dy = 0$.
3. $(1 + 4xy + 2y^2) \ dx + (1 + 4xy + 2x^2) \ dy = 0$. **(Bundelkhand 2004)**
4. $x \ dx + y \ dy = a^2 \dfrac{x \ dy - y \ dx}{x^2 + y^2}$. **(Agra 2006; Meerut 08; Gorakhpur 11; Bundelkhand 11)**
5. $[1 + e^{x/y}] \ dx + e^{x/y} [1 - (x/y)] \ dy = 0$. **(Meerut 2004B, 06)**
6. $(e^y + 1) \cos x \ dx + e^y \sin x \ dy = 0$. **(Agra 2006)**
7. $(\sin x \cos y + e^{2x}) \ dx + (\cos x \sin y + \tan y) \ dy = 0$.
8. $(y \sin 2x) \ dx - (1 + y^2 + \cos^2 x) \ dy = 0$.
9. $[y \{1 + (1/x)\} + \cos y] \ dx + [x + \log x - x \sin y] \ dy = 0$.

Answers 7

1. $x^2 - y^2 - xy + 5y = c$.
2. $ayx^2 + bxy^2 = c$
3. $x + 2x^2 y + 2xy^2 + y = c$
4. $x^2 + y^2 + 2a^2 \tan^{-1} (x/y) = c$

5. $x + y\, e^{x/y} = c$

6. $(e^y + 1) \sin x = c$

7. $\frac{1}{2} e^{2x} - \cos x \cos y + \log \sec y = c$

8. $y \cos 2x + 2y + \frac{2}{3} y^3 = c$

9. $y\,(x + \log x) + x \cos y = c$

3 Integrating Factors

(Lucknow 2007)

A differential equation which is not exact can sometimes be made exact by multiplying by some suitable function of x and y. Such a function is called an **integrating factor** (I.F.) of the equation.

Methods used to find the integrating factors:

Method 1: Integrating factor found by inspection: Sometimes integrating factors are found by inspection.

The students should **remember** the following exact differentials. These help us in finding the integrating factors.

(i) $d\,(xy) = x\,dy + y\,dx$,

(ii) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$,

(iii) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$,

(iv) $d\left(\frac{x^2}{y}\right) = \frac{2\,yx\,dx - x^2\,dy}{y^2}$,

(v) $d\left(\frac{y^2}{x}\right) = \frac{2xy\,dy - y^2\,dx}{x^2}$,

(vi) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{x^2 + y^2}$,

(vii) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}$,

(viii) $d\left(\log\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{xy}$,

(ix) $d\left(\log\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{xy}$,

(x) $d\left(\frac{e^x}{y}\right) = \frac{ye^x\,dx - e^x\,dy}{y^2}$,

(xi) $d\,\{\log \sqrt{(x^2 + y^2)}\} = \frac{x\,dx + y\,dy}{x^2 + y^2}$.

Illustrative Examples

Example 30: *Solve* $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$.

Solution: The given equation can be written as

$$(y\,dx + x\,dy) + xy^2\,dx - x^2\,y\,dy = 0,$$

or $$d\,(yx) + xy^2\,dx - x^2 y\,dy = 0\,.$$ [From (i) of article 3]

Dividing out by $x^2 y^2$, we have

$$\frac{d\,(yx)}{x^2\,y^2} + \frac{1}{x}\,dx - \frac{1}{y}\,dy = 0\,. \qquad \ldots(1)$$

Now putting $xy = v$ in (1), we have

$$(1/v^2)\,du + (1/x)\,dx - (1/y)\,dy = 0\,.$$

Now integrating each term, we get

$$-(1/v) + \log x - \log y = c, \text{ where } c \text{ is a constant}$$

or $\quad -(1/xy) + \log x - \log y = c \qquad [\because\ v = xy]$

or $\quad \log\,(x/y) = c + (1/xy)$, is the required solution.

Example 31: *Solve* $y\,dx - x\,dy + (1 + x^2)\,dx + x^2\,\sin\,y\,dy = 0\,.$ **(Lucknow 2005)**

Solution: Dividing each term by x^2, the given equation becomes

$$\frac{y\,dx - x\,dy}{x^2} + \left(\frac{1}{x^2} + 1\right)dx + \sin\,y\,dy = 0$$

or $$-\frac{x\,dy - y\,dx}{x^2} + \left\{\frac{1}{x^2} + 1\right\}dx + \sin\,y\,dy = 0$$

or $$-\,d\left\{\frac{y}{x}\right\} + \left\{\frac{1}{x^2} + 1\right\}dx + \sin\,y\,dy = 0\,, \qquad [\text{ Refer article 3, result (iii)}]$$

Integrating each term, we have

$$-\frac{y}{x} - \frac{1}{x} + x - \cos\,y = -\,c$$

or $$\frac{y}{x} + \frac{1}{x} - x + \cos\,y = c.$$

Comprehensive Exercise 8

Solve the following differential equations :

1. $x\,dy - y\,dx = xy^2\,dx\,.$

2. $x\,dy - y\,dx + 2x^3\,dx = 0.$

3. $(y^2e^x + 2xy)\,dx - x^2dy = 0\,.$ **(Gorakhpur 2008)**

4. $y\,(axy + e^x)\,dx - e^x\,dy = 0\,.$

5. $y\,(2x^2\,y + e^x)\,dx - (e^x + y^3)\,dy = 0\,.$ **(Gorakhpur 2009; Avadh 09)**

6. $y\sin 2x\,dx = (1 + y^2 + \cos^2\,x)\,dy\,.$

7. $x\,dy - y\,dx = (x^2 + y^2)\,dx\,.$

8. $x\,dx + y\,dy + (x^2 + y^2)\,dy = 0\,.$

9. $(x^2 + y^2 + a^2)\,y\,dy + (x^2 + y^2 - a^2)\,x\,dx = 0\,.$ **(Rohilkhand 2006, 07)**

Answers 8

1. $yx^2 + 2x = 2cy$.
2. $y + x^3 = cx$.
3. $e^x + (x^2/y) = c$.
4. $\frac{1}{2}ax^2 + (e^x/y) = c$.
5. $\frac{2}{3}x^3 - \frac{1}{2}y^2 + (e^x/y) = c$.
6. $-y\cos^2 x = y + \frac{1}{3}y^3 + c$.
7. $\tan^{-1}(y/x) = x + c$.
8. $x^2 + y^2 = e^{c-2y}$.
9. $(x^2 + y^2)^2 + 2a^2(y^2 - x^2) = c$.

Method 2: If the equation $M\,dx + N\,dy = 0$ is of the form

$$[f(x, y)]\,y\,dx + [F(x, y)]\,x\,dy = 0,$$

and $\quad Mx - Ny \neq 0$, then

$1/(Mx - Ny)$ is an integrating factor.

Note: If $Mx - Ny = 0$, then $Mx = Ny$ or $M/y = N/x$

i.e., the differential equation $M\,dx + N\,dy = 0$ reduces to $y\,dx + x\,dy = 0$ whose solution is $xy = c$.

Illustrative Examples

Example 32: *Solve* $(x^3y^3 + x^2y^2 + xy + 1)\,y\,dx + (x^3y^3 - x^2y^2 - xy + 1)\,x\,dy = 0.$

Solution: Here

$$Mx - Ny = (x^3y^3 + x^2y^2 + xy + 1)\,y \cdot x - (x^3y^3 - x^2y^2 - xy + 1)\,xy$$
$$= 2(x^3y^3 + x^2y^2) = 2x^2y^2(xy + 1).$$

$\therefore \quad$ I.F. $= 1/\{2x^2y^2(xy + 1)\}$.

The given equation may be written as

$$[x^2y^2(xy + 1) + (xy + 1)]\,y\,dx + [(x^3y^3 + 1) - xy(xy + 1)]\,x\,dy = 0.$$

Now multiplying by I.F. this becomes

$$\frac{[x^2y^2(xy + 1) + (xy + 1)]\,y\,dx}{2x^2y^2(xy + 1)} + \frac{[(xy + 1)\{(x^2y^2 - xy + 1) - xy\}]\,x\,dy}{2x^2y^2(xy + 1)} = 0$$

or $$\left[\frac{x^2y^2 + 1}{x^2y^2}\right]y\,dx + \left[\frac{(x^2y^2 - 2xy + 1)}{x^2y^2}\right]x\,dy = 0$$

or $$(y\,dx + x\,dy) + \frac{y\,dx + x\,dy}{x^2y^2} - \frac{2x^2y}{x^2y}\,dy = 0$$

or $$d\,(xy)+\frac{d\,(xy)}{x^2y^2}-\frac{2}{y}\,dy=0\,.$$

Now integrating each term, we get

$$xy+(-1/xy)-2\log y=c.$$

Comprehensive Exercise 9

Solve the following differential equations :

1. $(xy^2+2x^2y^3)\,dx+(x^2y-x^3y^2)\,dy=0\,.$
2. $(x^2y^2+xy+1)\,y\,dx+(x^2y^2-xy+1)\,x\,dy=0\,.$
3. $(xy\sin xy+\cos xy)\,y\,dx+(xy\sin xy-\cos xy)\,x\,dy=0\,.$ **(Garhwal 2009)**

Answers 9

1. $x^2=cy\,e^{1/(xy)}$
2. $xy+\log x-\log y-1/(xy)=c$
3. $x\sec(xy)=cy$

Method 3: When $Mx+Ny\neq 0$ and the equation is homogeneous, then the integrating factor of $M\,dx+N\,dy=0,$ is $1/(Mx+Ny)$.

Illustrative Examples

Example 33: *Solve* $x^2y\,dx-(x^3+y^3)\,dy=0\,.$

Solution: Here $Mx+Ny=x^3y-x^3y-y^4=-y^4\neq 0$

and the equation is homogeneous.

$\therefore$ I.F. $=1/(Mx+Ny)=-1/y^4.$

Multiplying the given equation by the I.F., we get

$$-\frac{x^2}{y^3}\,dx+\frac{x^3+y^3}{y^4}\,dy=0,\quad\text{or}\quad\frac{dy}{y}=\frac{x^2}{y^3}\,dx-\frac{x^3}{y^4}\,dy$$

or $$\frac{dy}{y}=\frac{1}{3}\,d\left(\frac{x^3}{y^3}\right).$$

$\therefore$ Integrating each term, we get

$$\log y = \frac{1}{3}\frac{x^3}{y^3} + \log c$$

or $$y = ce^{x^3/3y^3}.$$

Method 4: If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ is a function of x alone, say $f(x)$, then the integrating factor for $M\,dx + N\,dy = 0$ is $e^{\int f(x)\,dx}$.

Illustrative Examples

Example 34: *Solve* $(y + \frac{1}{3}y^3 + \frac{1}{2}x^2)\,dx + \frac{1}{4}(x + xy^2)\,dy = 0$. **(Lucknow 2007)**

Solution: Here $M = y + \frac{1}{3}y^3 + \frac{1}{2}x^2$ and $N = \frac{1}{4}(x + xy^2)$.

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + y^2 \text{ and } \frac{\partial N}{\partial x} = \frac{1}{4}(1 + y^2).$$

$$\therefore \quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4}{x(1+y^2)}\{(1+y^2) - \frac{1}{4}(1+y^2)\}$$

$$= \frac{4}{x(1+y^2)} \cdot \frac{3}{4}(1+y^2) = \frac{3}{4},$$

which is a function of x alone, say $f(x)$.

$$\therefore \quad \text{I.F.} = e^{\int f(x)\,dx} = e^{\int (3/x)\,dx} = e^{3\log x} = e^{\log x^3} = x^3.$$

Multiplying the given diff. equation by the I.F. x^3, we get

$$(x^3 y + \frac{1}{3}x^3 y^3 + \frac{1}{2}x^5)\,dx + \frac{1}{4}(x^4 + x^4 y^2)\,dy = 0,$$

which is an exact diff. equation and its solution is

$$\int M\,dx \text{ (treating } y \text{ as a constant)}$$

$$+ \int N\,dy \text{ (taking in } N \text{ terms without } x) = c$$

i.e., $$\frac{x^4 y}{4} + \frac{x^4 y^3}{12} + \frac{x^6}{12} = c,$$

or $$3x^4 y + x^4 y^3 + x^6 = 12c = k.$$

Method 5: If $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ is a function of y alone, say $f(y)$, then the integrating factor for $M\,dx + N\,dy = 0$ is $e^{\int f(y)\,dy}$.

Illustrative Examples

Example 35: *Solve* $(xy^3 + y)\,dx + 2\,(x^2 y^2 + x + y^4)\,dy = 0$. **(Lucknow 2006)**

Solution: Here $M = xy^3 + y$ and $N = 2\,(x^2 y^2 + x + y^4)$.

$$\therefore \qquad \frac{\partial M}{\partial y} = 3xy^2 + 1 \text{ and } \frac{\partial N}{\partial x} = 2\,(2xy^2 + 1)\,.$$

$$\therefore \qquad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy^3 + y}\{4xy^2 + 2 - 3xy^2 - 1\}$$

$$= \frac{1}{y\,(xy^2 + 1)}(xy^2 + 1)$$

$$= \frac{1}{y},$$

which is a function of y alone, say $f\,(y)$.

$$\therefore \qquad \text{I.F.} = e^{\int f\,(y)\,dy} = e^{\int (1/y)\,dy} = e^{\log y} = y.$$

Multiplying the given equation with the I.F. y, we have

$$(xy^4 + y^2)\,dx + 2\,(x^2 y^3 + xy + y^5)\,dy = 0,$$

which is an exact differential equation and its solution is

$$\int M\,dx \text{ (treating } y \text{ as a constant)}$$

$$+ \int N\,dy \text{ (taking in } N \text{ only those terms which do not contain } x\text{)}$$

$$= c, \text{ (a constant)}$$

i.e.,
$$\int (xy^4 + y^2)\,dx\,(y \text{ constant}) + \int 2\,y^5\,dy = c$$

or
$$\frac{1}{2}x^2 y^4 + xy^2 + \frac{1}{3}y^6 = c\,.$$

Comprehensive Exercise 10

Solve the following differential equations :

1. $(x^2 y - 2xy^2)\,dx - (x^3 - 3x^2 y)\,dy = 0$. **(Lucknow 2011)**
2. $(x^2 + y^2)\,dx - 2xy\,dy = 0$. **(Gorakhpur 2008)**
3. $(3x^2 y^4 + 2xy)\,dx + (2x^3 y^3 - x^2)\,dy = 0$.
4. $(xy^2 - x^2)\,dx + (3x^2 y^2 + x^2 y - 2x^3 + y^2)\,dy = 0$.

Answers 10

1. $(x/y)-2\log x+3\log y=c$

2. $x^2-y^2=cx$

3. $x^3y^2+(x^2/y)=c$

4. $e^{6y}[x^2(\frac{1}{2}y^2-\frac{1}{3}x)+(\frac{1}{6}y^2-\frac{1}{18}y+\frac{1}{108})]=0$

Method 6: If the equation is of the form

$$x^a y^b(my\,dx+nx\,dy)+x^c y^d(py\,dx+qx\,dy)=0, \qquad \ldots\text{(A)}$$

where a, b, m, n, c, d, p, q are constants, then the *integrating factor* is $x^h y^k$, where h, k can be obtained by applying the condition that after multiplication by $x^h y^k$, the equation (A) must become exact.

Example 36: *Solve* $(y^2+2x^2y)\,dx+(2x^3-xy)\,dy=0$.

Solution: The given equation can be written as

$$y(y\,dx-x\,dy)+2x^2(y\,dx+x\,dy)=0.$$

This equation is of the form (A), as mentioned above. So let the possible integrating factor be $x^h y^k$.

Multiplying the given equation by the proposed I.F. $x^h y^k$, we have

$$(x^h y^{k+2}+2x^{h+2}y^{k+1})\,dx+(2x^{h+3}y^k-x^{h+1}y^{k+1})\,dy=0. \qquad \ldots(1)$$

Now $\quad (\partial M/\partial y)=(k+2)\,x^h y^{k+1}+2(k+1)\,x^{h+2}\,y^k$

and $\quad (\partial N/\partial x)=2(h+3)\,x^{h+2}\,y^k-(h+1)\,x^h y^{k+1}$.

If the equation (1) is exact, then

$$(\partial M/\partial y)=(\partial N/\partial x).$$

∴ equating the coefficients of $x^h y^{k+1}$ and $x^{h+2}\,y^k$ on both sides, we get

$k+2=-h-1$ *i.e.*, $h+k+3=0$

and $2k+2=2h+6$ *i.e.*, $h-k+2=0$.

Solving these, we get $h=-5/2$ and $k=-1/2$.

∴ the integrating factor is $x^h y^k=x^{-5/2}y^{-1/2}$.

With these values of h and k the equation (1) becomes

$$(x^{-5/2}y^{3/2}+2x^{-1/2}y^{1/2})\,dx+(2x^{1/2}y^{-1/2}-x^{-3/2}y^{1/2})\,dy=0.$$

This differential equation is of the form $M\,dx+N\,dy=0$ and is exact as can be seen by verifying the condition $\partial M/\partial y=\partial N/\partial x$.

Hence its solution is

$$\int M\,dx \text{ (treating } y \text{ as a constant)}$$

$$+\int N\,dy \text{ (taking in } N \text{ only those terms which do not contain } x)$$

$$= c, \text{ (a constant)}$$

i.e., $$-\frac{2}{3}x^{-3/2}y^{3/2}+4x^{1/2}y^{1/2}+0=c.$$

Hence $-\frac{2}{3}x^{-3/2}y^{3/2}+4x^{1/2}y^{1/2}=c$ is the required solution.

(Gorakhpur 2007)

Comprehensive Exercise 11

Solve the following differential equations :

1. $(2y\,dx+3x\,dy)+2xy\,(3y\,dx+4x\,dy)=0$.
2. $(3x+2y^2)\,y\,dx+2x\,(2x+3y^2)\,dy=0$.
3. $x\,(3y\,dx+2x\,dy)+8y^4\,(y\,dx+3x\,dy)=0$. **(Lucknow 2010)**

Answers 11

1. $x^2y^3+2x^3y^4=c$ 2. $x^3y^4+x^2y^6=c$ 3. $x^3y^2+4y^6x^2=c$

4 Change of Variables

In some cases a suitable substitution (change of variable) reduces a given differential equation to one or the other of the forms already discussed and hence the equation can be solved.

Illustrative Examples

Example 37: *Solve* $\frac{x\,dx+y\,dy}{x\,dy-y\,dx}=\sqrt{\left(\frac{a^2-x^2-y^2}{x^2+y^2}\right)}$.

(Rohilkhand 2005; Purvanchal 09; Avadh 09; Bundelkhand 03; Garhwal 10)

Solution: Put $x=r\cos\theta$ and $y=r\sin\theta$. Then

$$x^2 + y^2 = r^2, \qquad \text{...(1)}$$

and $$y/x = \tan\theta. \qquad \text{...(2)}$$

Differentiating (1), we get

$$2x\,dx + 2y\,dy = 2r\,dr \quad \text{or} \quad x\,dx + y\,dy = r\,dr.$$

Differentiating (2), we get

$$\frac{x\,dy - y\,dx}{x^2} = \sec^2\theta\,d\theta$$

or $$x\,dy - y\,dx = x^2\sec^2\theta\,d\theta = r^2\cos^2\theta\sec^2\theta\,d\theta = r^2\,d\theta.$$

Substituting these values in the given differential equation, we get

$$\frac{r\,dr}{r^2\,d\theta} = \sqrt{\left(\frac{a^2 - r^2}{r^2}\right)} = \frac{\sqrt{(a^2 - r^2)}}{r}.$$

$$\therefore \quad \frac{dr}{d\theta} = \sqrt{(a^2 - r^2)}$$

or $$\frac{dr}{\sqrt{(a^2 - r^2)}} = d\theta. \qquad \text{[Separating the variables]}$$

Integrating, we get

$$\sin^{-1}(r/a) = \theta + c$$

or $$r = a\sin(\theta + c)$$

or $$\sqrt{(x^2 + y^2)} = a\sin\{\tan^{-1}(y/x) + c\}.$$

$$[\because r = \sqrt{(x^2 + y^2)} \text{ and } \theta = \tan^{-1}(y/x)]$$

5 Geometrical Problems

Remember the following formulae of the differential calculus:

Length of the tangent $= y\sqrt{\{1 + (dx/dy)^2\}}$.

Length of the subtangent $= y\,(dx/dy) = y/(dy/dx)$.

Length of the normal $= y\sqrt{\{1 + (dy/dx)^2\}}$.

Length of the subnormal $= y\,(dy/dx)$.

Intercept made by the tangent on the x-axis is $x - y\,(dx/dy)$, and on the y-axis is $y - x\,(dy/dx)$.

Angle ϕ between the radius vector and the tangent is given by

$$\tan\phi = r\,(d\theta/dr) \quad \text{or} \quad \cot\phi = (1/r)(dr/d\theta).$$

Length of the polar subtangent $= r^2\,(d\theta/dr)$.

Length of the polar subnormal $= (dr/d\theta)$.

Radius of curvature $\rho = [1 + (dy/dx)^2]^{3/2} / (d^2y/dx^2)$.

Illustrative Examples

Example 38: *Find the curves in which the polar subnormal is of constant length.*

Solution: Let the constant length of the polar subnormal be a.

But the length of the polar subnormal $= dr/d\theta$.

$\therefore$ as given in the question, $dr/d\theta = a$.

This is the differential equation of the required curves.

To solve, put it into the form $dr = a\,d\theta$. [Separating the variables]

$\therefore$ integrating, we get $r = a\theta + c$, where c is an arbitrary constant.

This is the polar equation of the required family of curves.

Example 39: *Find the curves for which the sum of the reciprocals of the radius vector and the polar subtangent is constant.*

Solution: The polar subtangent $= r^2\,(d\theta/dr)$.

$\therefore$ as given in the question, $\dfrac{1}{r} + \dfrac{1}{r^2}\dfrac{dr}{d\theta} = \lambda$, where λ is a constant.

This is the differential equation of the required curves. This equation may be written as

$$\frac{dr}{d\theta} = r^2\left(\lambda - \frac{1}{r}\right) = r(\lambda r - 1)$$

or $$d\theta = \frac{dr}{r(r\lambda - 1)} = \left(\frac{\lambda}{\lambda r - 1} - \frac{1}{r}\right)dr. \qquad \text{[Separating the variables]}$$

$\therefore$ integrating, we have

$$\theta + c = \log(\lambda r - 1) - \log r, \text{ where } c \text{ is an arbitrary constant}$$

or $$\theta + c = \log\{(\lambda r - 1)/r\}, \quad \text{or} \quad (\lambda r - 1)/r = e^{\theta + c}.$$

Hence $\lambda r - 1 = re^{\theta + c}$ is the equation of the required family of curves.

Example 40: *Find the curves in which the cartesian subtangent varies as the abscissa.*

Solution: The cartesian subtangent is $y/(dy/dx)$.

$\therefore$ as given in the question $y/(dy/dx) = \lambda x$, where λ is a constant

or $\lambda\,dx = \lambda x\,dy$

or $(dx/x) = \lambda\,(dy/y)$, separating the variables.

$\therefore$ integrating, we have

$$\log x = \lambda \log y + \log c, \text{ where } c \text{ is an arbitrary constant}$$

Hence $\log x = \log(y^{\lambda}.c)$

or $x = cy^{\lambda}$, is the required family of curves.

Example 41: *Show that the curve in which the angle between the tangent and the radius vector at every point is half of the vectorial angle is a cardioid.*

Solution: As given, $\phi = \frac{1}{2}\theta$, where θ is the vectorial angle.

$\therefore \qquad \tan\phi = \tan\frac{1}{2}\theta$.

But $\qquad \tan\phi = r\,(d\theta/dr)$.

$\therefore \qquad r\,(d\theta/dr) = \tan\frac{1}{2}\theta \quad$ or $\quad (dr/r) = (\cot\frac{1}{2}\theta)\,d\theta$,

in which the variables have been separated.

$\therefore$ integrating, we have $\log r = 2\log(\sin\frac{1}{2}\theta) + \log c$,

where c is an arbitrary constant

or $\qquad \log r = \log(\sin^2\frac{1}{2}\theta) + \log c, \quad$ or $\quad \log(r/c) = \log(\sin^2\frac{1}{2}\theta)$

or $\qquad (r/c) = \sin^2\frac{1}{2}\theta = \frac{1}{2}(1-\cos\theta), \quad$ or $\quad r = \frac{1}{2}c\,(1-\cos\theta)$

or $\qquad r = a\,(1-\cos\theta)$, where $a = \frac{1}{2}c$. This is a cardioid.

Example 42: *Find the curve for which the tangent at each point makes a constant angle* α *with the radius vector.*

Solution: As given $\phi = \alpha$ (a constant)

or $\qquad \tan\phi = \tan\alpha$.

But $\qquad \tan\phi = r\,(d\theta/dr)$.

$\therefore \qquad r\,(d\theta/dr) = \tan\alpha \quad$ or $\quad (dr/r) = (\cot\alpha)d\theta$.

Now the variables have been separated. Therefore integrating, we have

$\qquad \log r = \theta\cot\alpha + \log c$, where c is an arbitrary constant

or $\qquad \log r - \log c = \theta\cot\alpha$

or $\qquad \log(r/c) = \theta\cot\alpha$

or $\qquad r/c = e^{\theta\cot\alpha}$.

Hence $r = ce^{\theta\cot\alpha}$ is the required curve.

Example 43: *Show that the curve in which the slope of the tangent at any point equals the ratio of the abscissa to the ordinate of the point is a rectangular hyperbola.*

Solution: The slope of the tangent at any point $(x, y) = \tan\psi = dy/dx$.

$\therefore$ as given, $\frac{dy}{dx} = \frac{\text{abscissa}}{\text{ordinate}} = \frac{x}{y}$

or $\qquad y\,dy = x\,dx$, in which the variables have been separated.

$\therefore$ integrating, we have $\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$, where c is an arbitrary constant.

Hence $y^2 - x^2 = 2c = a^2$ (say) is the required curve which is the equation of a rectangular hyperbola.

Example 44: *Find the family of curves whose tangent forms an angle $\pi/4$ with the hyperbola $xy = c$.*

Solution: Let m_1 and m_2 be the gradients of the tangents of the required family of curves and the given hyperbola respectively. Since the angle between these tangents is given to be $\frac{1}{4}\pi$, therefore

$$\tan(\pi/4) = \frac{m_1 - m_2}{1 + m_1 m_2} \quad \text{or} \quad 1 = \frac{m_1 - m_2}{1 + m_1 m_2}$$

or $$1 + m_1 m_2 = m_1 - m_2 \quad \text{or} \quad m_1 = (1 + m_2)/(1 - m_2). \qquad \ldots(1)$$

Now $m_1 =$ gradient of the tangent of the required family of curves $= dy/dx$

and $m_2 = (dy/dx)$ for hyperbola $xy = c$.

$$\therefore \quad m_2 = \frac{d}{dx}\left(\frac{c}{x}\right) = -\frac{c}{x^2}, \qquad \left[\because y = \frac{c}{x}\right]$$

Hence from (1), we have

$$\frac{dy}{dx} = \frac{1 + (-c/x^2)}{1 - (-c/x^2)} = \frac{x^2 - c}{x^2 + c} = \left(1 - \frac{2c}{x^2 + c}\right)$$

or $$dy = \left(1 - \frac{2c}{x^2 + c}\right) dx,$$ in which the variables have been separated.

$\therefore$ integrating, we have $y = x - \frac{2c}{\sqrt{c}} \tan^{-1}\left(\frac{x}{\sqrt{c}}\right) + c_1$,

where c_1 is an arbitrary constant. This is the required family of curves.

Comprehensive Exercise 12

1. By the substitution $y^2 = v - x$ reduce the equation $y^3 (dy/dx) + x + y^2 = 0$ to the homogeneous form and hence solve the equation.
2. Find the curve in which the polar subtangent is constant.
3. Find the equation of the family of curves for which the sum of the reciprocals of the radius vector and the polar subnormal is constant.
4. Find the curve in which the cartesian sub-normal is equal to the abscissa.
5. Find the equation of the curve for which the cartesian subtangent varies as the reciprocal of the square of the abscissa.
6. Show that the parabola is the only curve in which the sub-normal is constant.

7. Show that the curve for which the normal at every point passes through a fixed point is a circle.
8. Find the equation of the curve in which the perpendicular from the origin on any tangent is equal to the abscissa of the point of contact.
9. The normal PN to a curve meets the x-axis in N. If the distance of N from the origin is twice the abscissa of P, prove that the curve is a rectangular hyperbola.
10. The tangent of any point P of a curve meets the x-axis in Q. If Q is on the positive side of the origin O and $OP = OQ$, show that the family of curves having this property are parabolas whose common axis is the x-axis.

Answers 12

1. $\tan^{-1}\{(y^2 + x)/x\} - \frac{1}{2}\log\{x^2 + (y^2 + x)^2\} = \log c$
2. $r(\theta + c) + a = 0$
3. $\theta = \lambda r - \log r + c$
4. $y^2 - x^2 = c^2$
5. $\lambda \log y = \frac{1}{3}x^3 + c$
6. $y^2 = 2\lambda\{x + (c/2\lambda)\}$
8. $y^2 + x^2 = cx$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The order of the differential equation

$$\frac{d^4 y}{dx^4} - 3\left(\frac{d^3 y}{dx^3}\right)^2 + 4\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \text{ is}$$

(a) 3
(b) 6
(c) 4
(d) 2

2. The integrating factor of the differential equation $\cos^2 x \frac{dy}{dx} + y = \tan x$ is

(a) $\sec^2 x$
(b) $e^{\tan x}$
(c) $\cos^2 x$
(d) $\tan x$

(Garhwal 2011)

3. The integrating factor of the differential equation $\frac{dy}{dx} + y \cot x = 2\cos x$ is

(a) $\sin x$
(b) $\log \sin x$
(c) $\cot x$
(d) $\cos x$

(Kumaun 2007)

4. The differential equation $M\,dx + N\,dy = 0$, where M and N are functions of x and y, is exact if

(a) $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ (b) $\dfrac{\partial M}{\partial x} = \dfrac{\partial N}{\partial y}$

(c) $\dfrac{\partial M}{\partial y} + \dfrac{\partial N}{\partial x} = 0$ (d) $\dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} = 0$

(Avadh 2005; Kumaun 06, 10, 11, 13; Kanpur 16)

5. The complete solution of the differential equation $(x^2 - ay)\,dx - (ax - y^2)\,dy = 0$ is

(a) $x^3 + y^3 + 3axy = c$ (b) $x^3 + y^3 - 3axy = 4$

(c) $x^3 + y^3 - 3axy = c$ (d) $x^3 + y^3 - 3axy = 0$

6. The integrating factor of differential equation $\dfrac{dx}{dy} + P(y)\,x = Q(y)$ is

(a) $e^{\int P\,dy}$ (b) $e^{\int P\,dx}$

(c) $e^{\int Q\,dx}$ (d) $e^{\int -P\,dx}$ **(Bundelkhand 2007)**

7. The differential equation $(x^2 + y^2 + a^2)\,y\,\dfrac{dy}{dx} + x\,(x^2 + y^2 - a^2) = 0$ is

(a) variables separable (b) linear

(c) homogeneous (d) exact **(Rohilkhand 2007)**

8. The order and degree of differential equation $\left(\dfrac{d^2 y}{dx^2}\right)^2 + y = \left(\dfrac{dy}{dx}\right)^4$

(a) 2, 4 (b) 4, 4

(c) 2, 2 (d) 4, 2 **(Garhwal 2009, 13)**

9. The order and degree of the differential equation $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2} = k\,\dfrac{d^2 y}{dx^2}$

(a) 3, 1 (b) 3, 2

(c) 3, 3 (d) 2, 2 **(Garhwal 2010)**

10. The order and degree of the differential equation $\dfrac{d^4 y}{dx^4} + 2k\left(\dfrac{d^2 y}{dx^2}\right)^4 + k^4 y = 0$ is

(a) 4, 4 (b) 4, 2

(c) 4, 1 (d) 1, 4 **(Garhwal 2009)**

11. Order and degree of differential equation $y'' + 2\sin^{-1}(y') - 3y = 0$ is

(a) order $= 2$, degree $= 1$

(b) order $= 2$, degree does not exist

(c) order does not exist and degree exists

(d) none of these **(Garhwal 2012)**

12. Degree of the differential equation $\frac{d^4y}{dx^4} + 2k^2\left(\frac{d^2y}{dx^2}\right) + k^4y = 0$ is

(a) 4 (b) 2

(c) 1 (d) 0 **(Garhwal 2014)**

13. Degree of the differential equation $\left[\frac{d^2y}{dx^2}\right]^2 - \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = 0$ is

(a) 2 (b) 3

(c) 4 (d) 5

(Kumaun 2006, 14; Garhwal 15)

14. The integrating factor of the differential equation $(x + 2y^3)\,dy = y\,dx$ is

(a) $\frac{1}{x}$ (b) $\frac{1}{y}$

(c) $\frac{x}{y}$ (d) xy **(Garhwal 2009)**

15. The integrating factor in the differential equation $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$ is

(a) $\frac{1}{e^{\tan y}}$ (b) $e^{\tan y}$

(c) $e^{\tan^{-1} y}$ (d) $\frac{1}{e^{\tan^{-1} y}}$ **(Garhwal 2009)**

16. In differential equation $y(2xy + e^x)\,dx - e^x\,dy = 0$ the integrating factor is

(a) e^{-x} (b) e^x

(c) $e^{1/x}$ (d) $e^{-1/x}$

17. The solution of the differential equation $\frac{dy}{dx} + y = e^{-x}$ is

(a) $ye^{-x} = x + c$ (b) $ye^x = x - c$

(c) $ye^x = x + c$ (d) $y = xe^{-x}$ **(Garhwal 2005)**

18. Solution of the differential equation $x\,dy - y\,dx - 2x^3\,dx = 0$ is

(a) $y = x^3 + cx$ (b) $y + x^3 + cx = 0$

(c) $y + x^3 = c$ (d) $y = x^3 + c$ **(Garhwal 2006)**

19. The solution of the differential equation $\frac{dy}{dx} + 2xy = e^{-x^2}$ is

(a) $ye^{-x^2} = x + c$ (b) $ye^{x^2} = x + c$

(c) $xe^{x^2} = y + c$ (d) $xe^{y^2} = x + c$ **(Garhwal 2010)**

20. The integrating factor in the differential equation $(x^2 y - 2xy^2)\,dx + (3x^2 y - x^3)\,dy = 0$ is

(a) $\frac{1}{xy}$ (b) $\frac{1}{x^2 y}$

(c) $\frac{1}{x^2 y^2}$ (d) $\frac{1}{x^3 y^3}$ **(Garhwal 2010)**

21. The solution of the differential equation $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$ is

(a) $\sin y = (e^x + 1) + c$ (b) $\sin x = (e^x + 1) + c$

(c) $\sin x\,(e^y + 1) = c$ (d) none of these **(Garhwal 2010)**

22. The integrating factor in the differential equation $(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$ is

(a) $e^{\tan^{-1} x}$ (b) $\frac{1}{e^{\tan^{-1} x}}$

(c) $e^{\tan{-1} y}$ (d) 1 **(Garhwal 2010)**

23. Integrating factor of differential equation $x \log x \frac{dy}{dx} + y = 2 \log x$ is

(a) $\frac{1}{x}$ (b) $x \log x$

(c) $\frac{1}{\log x}$ (d) $\log x$ **(Garhwal 2012)**

24. Integrating factor of the differential equation $(x + 2y^3)\frac{dy}{dx} = y$ is

(a) y (b) $1/y$

(c) 1 (d) $-1/y$ **(Garhwal 2013)**

25. Solution of the differential equation $\frac{dy}{dx} + y^2 = 0$ is

(a) $y = \frac{1}{x + c}$ (b) $y = -\frac{x^3}{3} + c$

(c) $y = c\,e^x$ (d) $y = x^2$ **(Kumaun 2015)**

26. The degree of the differential equation $\frac{d^2 y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = 0$ is

(a) 1 (b) 2

(c) 3 (d) 4 **(Kumaun 2011)**

27. The differential equation $x\,dx + 2y^2\,dy = 0$ is

(a) exact (b) non-exact

(c) partial (d) homogeneous **(Kumaun 2014)**

28. Integrating factor for the differential equation $\frac{dy}{dx}+\frac{3}{x}y=\frac{\sin x}{x^3}$ will be

(a) x^2 (b) x^3

(c) x (d) $\log x$ **(Kumaun 2014)**

29. The order of the differential equation $\frac{d^2y}{dt^2}+\left(\frac{dy}{dt}\right)^3+y^4=e^{-t}$ is

(a) 1 (b) 2

(c) 3 (d) none of these **(Kumaun 2015)**

30. Which of the following is a solution of the differential equation $\frac{dx}{dt}+3x=0$?

(a) $x=3e^{-t}$ (b) $x=2e^{-3t}$

(c) $x=-\frac{3}{2}t^2$ (d) $x=3t^2$

(Kumaun 2015)

31. Solution of the differential equation $\frac{dy}{dx}=\frac{y}{x}$ will represent

(a) family of circles (b) family of straight lines

(c) family of hyperbolas (d) none of these **(Kumaun 2015)**

32. The differential equation $\frac{dy}{dx}+\left(\frac{x}{1-x^2}\right)y=x^2$ is

(a) linear (b) homogeneous

(c) exact (d) total **(Kumaun 2013)**

33. The differential equation $\left(\frac{dy}{dx}\right)^2+5y^{1/3}=x$ is

(a) linear of order 2

(b) non-linear of order 1 and degree 2

(c) non-linear of order 1 and degree 6

(d) none of these **(Kanpur 2016)**

34. The degree of differential equation $\frac{d^2y}{dx^2}+\sqrt{1+\frac{dy}{dx}}=0$ is

(a) 0 (b) 1

(c) 2 (d) 4 **(Kanpur 2016)**

35. The differential equation whose solution is of the form $y=A\cos x+B\sin x$ is

(a) $\frac{d^2y}{dx^2}+y=0$ (b) $\frac{d^2y}{dx^2}-y=0$

(c) $\frac{d^2y}{dx^2}+y=A\cos x$ (d) $\frac{d^2y}{dx^2}+y=B\sin x$ **(Kanpur 2016)**

36. The integrating factor of $(x+1)\dfrac{dy}{dx}-y=e^{3x}(1+x)^2$ is

(a) $x+1$ (b) $\log(x+1)$

(c) $\dfrac{1}{x+1}$ (d) e^{x+1} **(Kanpur 2016)**

37. The solution of $y'+y\tan x=\cos x,\ y(0)=0$ is

(a) $\cos x$ (b) $\sin x$

(c) $x\cos x$ (d) $x\sin x$ **(Kanpur 2016)**

38. Which one is the homogeneous equation?

(a) $\dfrac{dy}{dx}=\dfrac{x^2+2y^2}{3x^2+2y^3}$ (b) $\dfrac{dy}{dx}=\dfrac{x^3+2y^3}{3x^2+2y^2}$

(c) $\dfrac{dy}{dx}=\dfrac{x^2+2y^2}{3x^2+4y^2}$ (d) $\dfrac{dy}{dx}=\dfrac{x^2+2y^2}{3x^2+2y}$ **(Kanpur 2016)**

39. Solution of differential equation $3e^x\tan y\,dx+(1-e^x)\sec^2 y\,dy=0$ is

(a) $\tan y=c(1-e^x)^3$ (b) $\tan y=c(1-e^x)^2$

(c) $\sin y=c(e^x-1)^2$ (d) $\tan^{-1} y=c(1-e^x)^3$

(Kanpur 2016)

40. Solution of differential equation $\dfrac{dy}{dx}+\dfrac{3y}{x}=\dfrac{1}{x^4}$ is

(a) $yx^3=\log x+c$ (b) $yx^3=x\log x+c$

(c) $yx^{-3}=\log x+\dfrac{c}{x}$ (d) $yx^2=\log x+c$ **(Kanpur 2016)**

41. In solving $(2x-y+2)\,dx+(6x-3y+4)\,dy=0$ substitution used is

(a) $2x-y=z$ (b) $2x+y=z$

(c) $x-y=z$ (d) $x+y=z$ **(Kanpur 2016)**

42. The differential equation $(y^2e^{xy^2}+6x)\,dx+(2xye^{xy^2}-4y)\,dy=0$ is

(a) non-linear, non-homogeneous and not exact

(b) non-linear, non-homogeneous and exact

(c) non-linear, homogeneous and exact

(d) linear, homogeneous and exact

43. If $M\,dx+N\,dy=0$ and $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$ then differential equation is

(a) linear (b) homogeneous

(c) non-homogeneous (d) exact differential equation

44. The differential equation $M\,dx+N\,dy=0$ can be reduced to exact if $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)$ is

(a) a function of x and y (b) a function of x alone

(c) a function of y alone (d) none of these

45. The differential equation $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is

(a) a function of x and y (b) a function of x alone
(c) a function of y alone (d) none of these

46. The differential equation $(ay^2+x+x^8)\,dx+(y^8-y+bxy)\,dy=0$ is exact if

(a) $2a=b$ (b) $a=b$
(c) $2a\neq b$ (d) $a\neq b$

47. The integrating factor of $\frac{dy}{dx}+2xy=e^{-x^2}$ is

(a) e^{-x^2} (b) e^{x^2}
(c) e^{-x} (d) e^{x}

48. The integrating factor of $\cos^2 x\frac{dy}{dx}+y=\tan x$ is

(a) $e^{\tan x}$ (b) $e^{-\tan x}$
(c) $e^{\cot x}$ (d) $e^{-\cot x}$

49. The integrating factor of $\frac{dy}{dx}+\frac{1}{x}y=x^2y^6$ is

(a) x^5 (b) x^{-5}
(c) $\frac{1}{x^5}$ (d) $\frac{1}{x^{-5}}$

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. The order of the differential equation

$$\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=0 \text{ is ....}$$

2. The degree of the differential equation $\left(\frac{dy}{dx}\right)^3-4xy\frac{dy}{dx}+8y^2=0$ is ....

3. The order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2-\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=0$ is

...... and ...... **(Rohilkhand 2006)**

4. A differential equation of the form $\frac{dy}{dx}=\frac{f_1(x,y)}{f_2(x,y)}$, where $f_1(x,y)$ and $f_2(x,y)$ are homogeneous functions of x and y of the same degree, is called a ....

5. To solve the differential equation $x^2y\,dx-(x^3+y^3)\,dy=0$, we put...... .

6. To solve the differential equation $\frac{dy}{dx}=\frac{2x+y+1}{4x+2y-1}$, we put ....

7. A differential equation is said to be ...... when the dependent variable y and all its derivatives occur in the first degree only and are not multiplied together.

8. The integrating factor of the differential equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only, is …… .

9. The integrating factor of the differential equation $(x^2 - 1)\frac{dy}{dx} + 2xy = 1$ is …… .

10. The integrating factor of the differential equation $\frac{dy}{dx} + 2y \tan x = \sin x$ is …… .
(Kumaun 2009)

11. The ordinary differential equation $M\,dx + N\,dy = 0$, where M and N are functions of x and y, is exact if and only if …… .
(Meerut 2003; Bundelkhand 08; Rohilkhand 10)

12. A differential equation which is not exact can sometimes be made exact by multiplying by some suitable function of x and y. Such a function is called an …… of the equation.

13. The differential equation $\frac{dx}{dy} + \frac{1}{1+y^2} \cdot x = \frac{\tan^{-1} y}{1+y^2}$ is linear with …… as dependent variable. **(Kumaun 2010)**

14. If the differential equation $M\,dx + N\,dy = 0$ is homogeneous and if $Mx + Ny \neq 0$, then the integrating factor of this differential equation is …… .
(Bundelkhand 2007)

15. The differential equation $(x + y)\,dy = a^2\,dx$ is linear with …… as independent variable.

16. A solution of differential equation of three order has …… constants.
(Agra 2007)

17. The differential equation $y^2\,dx + (xy + x^2)\,dy = 0$ is …… . **(Kumaun 2007)**

18. Differential equation $\frac{dy}{dx} + Py = Q$ is …… . **(Kumaun 2008)**

True or False

Write 'T' for true and 'F' for false statement.

1. The differential equation $y^2\,dx + (xy + x^2)\,dy = 0$ is homogeneous.

2. The order of the differential equation $\frac{d^3 y}{dx^3} - 7\left(\frac{dy}{dx}\right)^2 + 6y = 0$ is 2.

3. The degree of the differential equation $\frac{d^2 y}{dx^2} + 4\left(\frac{dy}{dx}\right)^2 - 5y = 0$ is 2.

4. The differential equation $x^2 \frac{dy}{dx} + y = 1$ is linear with y as dependent variable.

5. The integrating factor of the differential equation $\frac{dy}{dx} + y \sec x = \tan x$ is $(\sec x + \tan x)$.

6. The differential equation $(x + y + 1)\dfrac{dy}{dx} = 1$ is linear with y as dependent variable.

7. The differential equation $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$ is exact.

8. The integrating factor of the differential equation $\dfrac{dy}{dx} + \dfrac{y}{x} = \sin x^2$ is $\log x$.

9. The differential equation $\dfrac{dy}{dx} + \dfrac{1}{x}\,y = x^2 y^6$ is linear with y as dependent variable.

10. The differential equation $x\dfrac{dy}{dx} + y = x^2 y^4$ can be made linear by putting $\dfrac{1}{y^3} = v$.

11. A differential equation of the form $\dfrac{dy}{dx} + Py = Qy^n$,

where P and Q are functions of x alone, is called Bernoulli's equation.

(Meerut 2003)

12. If the differential equation $M\,dx + N\,dy = 0$ is exact, its solution is

$$\underset{\text{treating } y \text{ as constant}}{\int M\,dx} + \underset{\substack{\text{taking only those}\\ \text{terms in } N \text{ which}\\ \text{do not contain } x}}{\int N\,dy} = c.$$

(Meerut 2003)

Answers

Multiple Choice Questions

1.	(c)	**2.**	(b)	**3.**	(a)	**4.**	(a)	**5.**	(c)
6.	(a)	**7.**	(d)	**8.**	(c)	**9.**	(d)	**10.**	(c)
11.	(b)	**12.**	(c)	**13.**	(a)	**14.**	(b)	**15.**	(c)
16.	(b)	**17.**	(c)	**18.**	(a)	**19.**	(b)	**20.**	(c)
21.	(c)	**22.**	(c)	**23.**	(d)	**24.**	(b)	**25.**	(a)
26.	(a)	**27.**	(a)	**28.**	(b)	**29.**	(b)	**30.**	(b)
31.	(b)	**32.**	(a)	**33.**	(c)	**34.**	(c)	**35.**	(a)
36.	(c)	**37.**	(c)	**38.**	(c)	**39.**	(a)	**40.**	(a)
41.	(a)	**42.**	(b)	**43.**	(d)	**44.**	(b)	**45.**	(c)
46.	(a)	**47.**	(b)	**48.**	(a)	**49.**	(c)		

Fill in the Blank(s)

1. 3
2. 3
3. 2, 2
4. homogeneous equation
5. $y = vx$
6. $2x + y = v$
7. linear
8. $e^{\int P\,dx}$
9. $x^2 - 1$
10. $\sec^2 x$
11. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
12. integrating factor
13. x
14. $\frac{1}{Mx + Ny}$
15. y
16. 3
17. homogeneous equation
18. linear differential equation

True or False

1. *T*
2. *F*
3. *F*
4. *T*
5. *T*
6. *F*
7. *T*
8. *F*
9. *F*
10. *T*
11. *T*
12. *T*

Chapter

2

Differential Equations of The First Order But Not of The First Degree

1 Introduction

In the present chapter we shall discuss the solutions of differential equations which are of the first order but are of degree higher than one. Such differential equations will contain only the first differential coefficient dy/dx but it will occur in a degree higher than one. It is usual to denote dy/dx by p. The general form of such a differential equation is then

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \ldots + A_{n-1} p + A_n = 0 ,$$

where $A_1 , A_2 , \ldots , A_n$ are some functions of x and y.

Now we shall consider the various methods of solving the differential equations of the above type.

2 Equations Solvable for p

Suppose a differential equation of first order and of degree n can be solved for p *i.e.*, it can be resolved into n linear factors in p of the type

$$\{p - f_1(x, y)\}\{p - f_2(x, y)\}\ldots\{p - f_n(x, y)\} = 0 .$$

We can equate each factor to zero and the resulting differential equations of the first order and first degree can be solved. Let their solutions be

$$\phi_1(x, y, c_1) = 0, \phi_2(x, y, c_2) = 0\ , \ldots, \phi_n(x, y, c_n) = 0,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants. There is no loss of generality if we replace the arbitrary constants $c_1, c_2, \ldots, c_n$ by a single arbitrary constant c because in any of the above n solutions c is free to take any real value. Thus the n solutions of the given differential equation are

$$\phi_1(x, y, c) = 0,\ \phi_2(x, y, c) = 0, \ldots, \phi_n(x, y, c) = 0\ .$$

Combining the above equations, we get a single composite solution as

$$\phi_1(x, y, c)\, \phi_2(x, y, c) \ldots \phi_n(x, y, c) = 0\ .$$

Illustrative Examples

Example 1: *Solve* $p^2 - 7p + 12 = 0$. **(Bundelkhand 2008; Kanpur 04)**

Solution: Resolving into linear factors, the given differential equation can be written as $(p - 3)(p - 4) = 0$.

Its component equations are $p = 3, p = 4$.

Solving the differential equation $p = 3$ *i.e.,* $dy / dx = 3$, we get $y = 3x + c$. Also the solution of the differential equation $p = 4$ is $y = 4x + c$.

So the solutions of the given differential equation are

$$y = 3x + c,\ y = 4x + c\ .$$

The single combined solution is

$$(y - 3x - c)(y - 4x - c) = 0\ .$$

Example 2: *Solve* $p^2 + 2py \cot x = y^2$. **(Gorakhpur 2005; Rohilkhand 09; Kanpur 08; Bundelkhand 04)**

Solution: The given differential equation is

$$p^2 + 2py \cot x - y^2 = 0\ .$$

Solving for p , we get

$$p = \frac{dy}{dx} = \frac{-2y \cot x \pm \sqrt{(4y^2 \cot^2 x + 4y^2)}}{2}$$

$$= -y \cot x \pm y \operatorname{cosec} x = y(-\cot x \pm \operatorname{cosec} x)\ .$$

Thus the component equations are

$$\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)\ , \qquad \ldots(1)$$

and $$\frac{dy}{dx} = -y(\cot x + \operatorname{cosec} x)\ . \qquad \ldots(2)$$

In each of the above differential equations, the variables are separable.

From (1), separating the variables, we have

$$\frac{dy}{y} = (-\cot x + \operatorname{cosec} x)\, dx\,.$$

Integrating, we get

$$\log y - \log c = -\log \sin x + \log \tan \frac{1}{2} x$$

or $$\log\left(\frac{y}{c}\right) = \log\left\{\frac{\tan \frac{1}{2} x}{\sin x}\right\} = \log\left\{\frac{\sin \frac{1}{2} x / \cos \frac{1}{2} x}{2 \sin \frac{1}{2} x \cos \frac{1}{2} x}\right\}$$

$$= \log\left\{\frac{1}{2\cos^2 \frac{1}{2} x}\right\} = \log\left\{\frac{1}{1+\cos x}\right\}.$$

∴ $$y/c = 1/(1+\cos x)$$

or $$y = c/(1+\cos x). \qquad \ldots(3)$$

From (2), separating the variables, we have

$$\frac{dy}{y} = -(\cot x + \operatorname{cosec} x)\, dx\,.$$

Integrating, we get

$$\log y - \log c = -(\log \sin x + \log \tan \frac{1}{2} x)$$

or $$\log(y/c) = -\log\{(\sin x)(\tan \frac{1}{2} x)\}$$

$$= -\log\{2 \sin \frac{1}{2} x \cos \frac{1}{2} x \cdot (\sin \frac{1}{2} x / \cos \frac{1}{2} x)\} = -\log(2 \sin^2 \frac{1}{2} x)$$

$$= -\log(1-\cos x) = \log(1-\cos x)^{-1} = \log\{1/(1-\cos x)\}\,.$$

∴ $$y/c = 1/(1-\cos x)$$

or $$y = c/(1-\cos x). \qquad \ldots(4)$$

Thus the solutions of the given differential equation are given by (3) and (4). The single combined solution is

$$\left(y - \frac{c}{1+\cos x}\right)\left(y - \frac{c}{1-\cos x}\right) = 0\,.$$

Example 3: *Solve* $(p - xy)(p - x^2)(p - y^2) = 0$.

Solution: Equating each factor to zero, the component equations are

$$p - xy = 0\,, \qquad \ldots(1)$$

$$p - x^2 = 0\,, \qquad \ldots(2)$$

and $$p - y^2 = 0\,. \qquad \ldots(3)$$

From (1), $p = xy$

or $dy/dx = xy$

or $(1/y)\, dy = x\, dx.$

Integrating, we get

$$\log y = \frac{1}{2} x^2 + \log c$$

or $$\log (y/c) = \frac{1}{2} x^2$$

or $$y/c = e^{\frac{1}{2}x^2}$$

or $$y = ce^{\frac{1}{2}x^2} . \qquad \ldots(4)$$

From (2),

$$p = x^2 \quad \text{or} \quad dy/dx = x^2 \quad \text{or} \quad dy = x^2 \, dx .$$

Integrating, we get

$$y = \frac{1}{3} x^3 + \frac{1}{3} c$$

or $$3y - x^3 = c . \qquad \ldots(5)$$

From (3), $p = y^2$

or $$dy/dx = y^2$$

or $$(1/y^2)\, dy = dx.$$

Integrating, we have

$$-(1/y) = x + c$$

or $$xy + cy + 1 = 0 . \qquad \ldots(6)$$

Now (4), (5) and (6) are the solutions of the given differential equation. The single combined solution is

$$\left(y - ce^{\frac{1}{2}x^2}\right)(3y - x^3 - c)\,(xy + cy + 1) = 0 .$$

Example 4: *Solve* $p\,(p - y) = x\,(x + y)$.

Solution: The given differential equation can be written as

$$p^2 - py - (x^2 + xy) = 0 .$$

Solving for p, we get

$$p = \frac{dy}{dx} = \frac{y \pm \sqrt{(y^2 + 4x^2 - 4xy)}}{2} = \frac{y \pm \sqrt{\{(y + 2x)^2\}}}{2} = \frac{y \pm (y + 2x)}{2} .$$

Thus the component equations are

$$dy/dx = \{y + (y + 2x)\}/2 = y + x, \qquad \ldots(1)$$

and $$dy/dx = (y - y - 2x)/2 = -x . \qquad \ldots(2)$$

From (1), we have $(dy/dx) - y = x$, which is a linear differential equation with y as the dependent variable.

Integrating factor is

$$e^{\int -dx} = e^{-x}.$$

Therefore its solution is

$$ye^{-x} = \int xe^{-x}dx + c$$

or $$ye^{-x} = -xe^{-x} - \int (-e^{-x})\, dx + c \text{ , integrating by parts}$$

or $$ye^{-x} = -xe^{-x} - e^{-x} + c$$

or $$ye^{-x} = -e^{-x}(x+1) + c$$

or $$y = -(x+1) + ce^{x}$$

or $$y + x + 1 - ce^{x} = 0 . \qquad \text{...(3)}$$

From (2), we have

$$dy = -x\, dx .$$

Integrating, we get

$$y = -\frac{1}{2}x^2 + \frac{1}{2}c$$

or $$2y + x^2 - c = 0 . \qquad \text{...(4)}$$

Thus (3) and (4) are the required solutions of the differential equation. The single combined solution is

$$(y + x + 1 - ce^{x})(2y + x^2 - c) = 0 .$$

Example 5: *Solve* $\left(1 - y^2 + \frac{y^4}{x^2}\right)p^2 - 2\frac{y}{x}p + \frac{y^2}{x^2} = 0$.

Solution: Multiplying throughout by x^2, the given differential equation becomes

$$(x^2 - x^2y^2 + y^4)\, p^2 - 2xyp + y^2 = 0$$

or $$x^2p^2 - 2xyp + y^2 = p^2(x^2y^2 - y^4)$$

or $$(xp - y)^2 = p^2y^2(x^2 - y^2)$$

∴ $$xp - y = \pm\, py\sqrt{(x^2 - y^2)}$$

or $$p[x \pm y\sqrt{(x^2 - y^2)}] = y$$

or $$\frac{1}{p} = \frac{dx}{dy} = \frac{x \pm y\sqrt{(x^2 - y^2)}}{y} .$$

Putting $x = vy$ so that $(dx/dy) = v + y\,(dv/dy)$, we have

$$v + y\frac{dv}{dy} = \frac{vy \pm \sqrt{(v^2y^2 - y^2)}}{y} = v \pm y\sqrt{(v^2 - 1)}$$

or $$y\frac{dv}{dy} = \pm\, y\sqrt{(v^2 - 1)}$$

or $$\frac{dv}{dy} = \pm\sqrt{(v^2 - 1)}$$

or $$dy = \pm[1/\sqrt{(v^2 - 1)}]\, dv, \text{ separating the variables.}$$

Integrating, we get $y + c = \pm \cosh^{-1} v$

or $\qquad y + c = \pm \cosh^{-1}(x / y)$

or $\qquad \cosh^{-1}(x/y) = \pm (y + c)$

or $\qquad x/y = \cosh\{\pm (y + c)\}$.

Since $\cosh(-x) = \cosh x$, therefore, we have

$\qquad x = y \cosh(y + c)$ and $x = y \cosh(y + c)$, as the required solutions.

The single combined solution is $[x - y \cosh(y + c)]^2 = 0$.

Comprehensive Exercise 1

Solve the following differential equations :

1. $p^2 - 5p + 6 = 0$. **(Rohilkhand 2006; Kanpur 02; Kumaun 12)**
2. $y^2 + xyp - x^2 p^2 = 0$. **(Meerut 2007B)**
3. $xyp^2 - (x^2 + y^2) p + xy = 0$. **(Gorakhpur 2011)**
4. $xyp^2 + p(3x^2 - 2y^2) - 6xp = 0$.
5. $x(dy/dx)^2 + (y - x)(dy/dx) - y = 0$. **(Meerut 2006B; Bundelkhand 09, 10)**
6. $y(dy/dx)^2 + (x - y)(dy/dx) - x = 0$. **(Meerut 2005B)**

Answers 1

1. $(y - 2x - c)(y - 3x - c) = 0$
2. $(y^2 - c x^{1+\sqrt{5}})(y^2 - c x^{1-\sqrt{5}}) = 0$
3. $(x^2 - y^2 - c)(y - cx) = 0$
4. $(y - cx^2)(y^2 + 3x^2 - c) = 0$
5. $(y - x - c)(xy - c) = 0$
6. $(y - x - c)(x^2 + y^2 - c^2) = 0$

3 Equations Solvable for y

Suppose the given differential equation is solvable for y. Then it can be put in the form

$$y = f(x, p). \qquad \ldots(1)$$

Differentiating (1) w.r.t. x and denoting dy / dx by p, we obtain

$$p = \phi\left(x, p, \frac{dp}{dx}\right), \qquad \ldots(2)$$

which is a differential equation in two variables x and p. Suppose it is possible to solve the differential equation (2). Let its solution be

$$F(x, p, c) = 0, \qquad \ldots(3)$$

where c is the arbitrary constant.

Eliminating p between (1) and (3), we get the required solution of (1) in the form

$$\psi(x, y, c) = 0.$$

If it is not easily practicable to eliminate p between (1) and (3), we may solve (1) and (3) to get x and y in terms of p and c in the form

$$x = f_1(p, c), \; y = f_2(p, c), \qquad \ldots(4)$$

which give us the required solution of (1) in the form of parametric equations, the parameter being p.

Special case: Equations that do not contain x: In this case the equation has the form $f(y, p) = 0$. If it is solvable for p, it will give

$$p = \phi(y) \qquad i.e., \qquad dy/dx = \phi(y),$$

which can be easily solved by separating the variables.

If it is solvable for y, it will give $y = \psi(p)$, which can be solved by the method just explained in article 3.

Illustrative Examples

Example 6: Solve $y + px = p^2 x^4$.

(Rohilkhand 2006; Avadh 08; Agra 08; Purvanchal 11; Lucknow 11; Garhwal 10)

Solution: Solving for y, the given differential equation can be written as

$$y = -px + p^2 x^4. \qquad \ldots(1)$$

Differentiating (1) w.r.t. x and denoting dy/dx by p, we get

$$p = -p - x\frac{dp}{dx} + 2px^4\frac{dp}{dx} + 4p^2x^3$$

or $$2p - 4p^2x^3 + x\frac{dp}{dx}(1 - 2x^3p) = 0$$

or $$2p(1 - 2x^3p) + x\frac{dp}{dx}(1 - 2x^3p) = 0$$

or $$(1 - 2x^3p)\left(2p + x\frac{dp}{dx}\right) = 0.$$

$\therefore$ $$2p + x\frac{dp}{dx} = 0, \qquad \ldots(2)$$

and $$1-2x^3p=0.$$

From (2), $$dp/dx=-2p/x$$

or $$dp/p=-(2/x)\,dx.$$

Integrating, we get

$$\log p=-2\log x+\log c$$

or $$\log px^2=\log c$$

or $$px^2=c \quad \text{or} \quad p=c/x^2.$$

Substituting this value of p in (1), we get

$$y=-x\,(c/x^2)+(c^2/x^4).x^4$$

or $$y=-(c/x)+c^2$$

or $xy=c^2x-c$, as the required solution of (1).

Note: If we eliminate p between (1) and (3), we get

$$y=-(1/2x^3)x+(1/4x^6).x^4=-(1/2x^2)+(1/4x^2)=-1/4x^2,$$

which is also a solution of (1) because it satisfies (1). This solution itself does not contain any arbitrary constant. Also it cannot be obtained from the general solution $xy=c^2x-c$ by giving any particular value to c. Such a solution is called the **singular solution** and we shall discuss it in details later on.

Example 7: *Solve* $y=2px-p^2$. **(Meerut 2001, 04B; Kanpur 07, 15; Kumaun 09, 11)**

Solution: The given differential equation is

$$y=2px-p^2. \qquad \ldots(1)$$

Differentiating (1) w.r.t. x and denoting dy/dx by p, we get

$$p=2p+2x\frac{dp}{dx}-2p\frac{dp}{dx}$$

or $$p+2\frac{dp}{dx}(x-p)=0$$

or $$p\frac{dx}{dp}+2x=2p$$

or $$\frac{dx}{dp}+\frac{2}{p}x=2, \qquad \ldots(2)$$

which is a linear differential equation with x as dependent variable and p as the independent variable.

$\therefore$ integrating factor $=e^{\int(2/p)dp}=e^{2\log p}=e^{\log p^2}=p^2$.

$\therefore$ the solution of (2) is $xp^2=\int 2p^2\,dp+c=\frac{2}{3}p^3+c$

or $$x=\frac{2}{3}p+cp^{-2} \qquad \ldots(3)$$

Here it is not easily practicable to eliminate p between (1) and (3). So putting the value of x from (3) in (1), we get $y = 2p\left(\frac{2}{3}p + cp^{-2}\right) - p^2$

or $$y = \frac{1}{3}p^2 + 2cp^{-1}. \qquad \text{...(4)}$$

The equations (3) and (4), which express x and y in terms of a parameter p, constitute the required solution of (1).

Example 8: *Solve* $y - x = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$.

(Meerut 2008; Kumaun 08)

Solution: Denoting dy/dx by p, the given differential equation can be written as

$$y - x = xp + p^2.$$

Solving it for y, we get $y = x(1 + p) + p^2$. ...(1)

Differentiating (1) w.r.t. x and writing p for dy/dx, we get

$$p = 1 + p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

or $$1 + x\frac{dp}{dx} + 2p\frac{dp}{dx} = 0$$

or $$\frac{dx}{dp} + x = -2p, \qquad \text{...(2)}$$

which is a linear differential equation.

Here the I.F. $= e^{\int 1\, dp} = e^p$.

$\therefore$ the solution of (2) is

$$x\, e^p = \int -2pe^p\, dp + c$$

or $$x\, e^p = -2(pe^p - e^p) + c, \text{ integrating by parts}$$

or $$x\, e^p = -2e^p(p - 1) + c$$

or $$x = -2(p - 1) + ce^{-p}. \qquad \text{...(3)}$$

Substituting this value of x in (1), we get

$$y = (1 + p)\{-2(p - 1) + ce^{-p}\} + p^2$$

or $$y = c(1 + p)e^{-p} + 2 - p^2. \qquad \text{...(4)}$$

The equations (3) and (4), which express x and y in terms of a parameter p, constitute the required solution of (1).

Example 9: Solve $y = a\sqrt{(1 + p^2)}$.

Solution: The given differential equation is

$$y = a\sqrt{(1 + p^2)}. \qquad \text{...(1)}$$

Differentiating (1) w.r.t. x and writing p for dy/dx, we get

$$p = a \cdot \frac{1}{2}(1 + p^2)^{-1/2} \cdot 2p(dp/dx) \quad \text{or} \quad \frac{a\, dp}{\sqrt{(1 + p^2)}} = dx.$$

Integrating, we get

$$x = a \log \{p + \sqrt{(1+p^2)}\} + c. \qquad \ldots(2)$$

Now from (1),

$$y^2 = a^2 + a^2 p^2 \quad \text{or} \quad a^2 p^2 = y^2 - a^2 \quad \text{or} \quad p = \sqrt{(y^2 - a^2)}/a\,.$$

Putting $\sqrt{(1+p^2)} = y/a$ and $p = \sqrt{(y^2 - a^2)}/a$ in (2), we get

$$x = a \log \left\{ \frac{\sqrt{(y^2 - a^2)}}{a} + \frac{y}{a} \right\} + c$$

or $$x = a \log \{y + \sqrt{(y^2 - a^2)}\} - a \log a + c$$

or $$x = a \log \{y + \sqrt{(y^2 - a^2)}\} + c,$$

writing c for $-a \log a + c$ because c is an arbitrary constant.

Hence the required solution is

$$x = \log \{y + \sqrt{(y^2 - a^2)}\} + c\,.$$

Comprehensive Exercise 2

Solve the following differential equations :

1. $y = 2px + p^4 x^2$. **(Meerut 2006; Avadh 09)**
2. $4p^3 + 3xp = y$.
3. $x^2 + p^2 x = yp$.
4. $y - 2xp = f\,(xp^2)$. **(Meerut 2007)**
5. $y = x\{p + \sqrt{(1+p^2)}\}$. **(Kanpur 2009; Rohilkhand 09)**
6. $y = \sin p - p \cos p.$ **(Gorakhpur 2006)**
7. By differentiating with respect to x the equation $p^3 + xp^2 = y$, obtain its general solution in the form $x = f(p), y = \phi(p)$.

Answers 2

1. $(y - c^2)^2 = 4cx$
2. $x = -\frac{12}{7} p^2 + cp^{-3/2}, y = -\frac{8}{7} p^3 + 3cp^{-1/2}$
3. $x = -\frac{1}{3} p^3 + cp^{1/2}, y = (-\frac{1}{3} p^2 + cp^{1/2})^2 / p + p^2 (-\frac{1}{3} p^2 + cp^{1/2})$
4. $y = 2c\sqrt{x} + f\,(c^2)$
5. $x^2 + y^2 - 2xc = 0$

6. $x = c - \cos p,\ y = \sin p - \cos p$

7. $x = \dfrac{\frac{1}{2}(2c + 3p^2 - 2p^3)}{(p-1)^2},\ y = \dfrac{2cp^2 + 2p^3 - p^4}{2\,(p-1)^2}$

4 Equations Solvable for x

Suppose the given differential equation is solvable for x. Then it can be put in the form

$$x = f(y, p). \qquad \ldots(1)$$

Differentiating (1) w.r.t. y and writing $1/p$ for dx/dy, we get

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right), \qquad \ldots(2)$$

which is a differential equation in two variables y and p. Suppose it is possible to solve the differential equation (2). Let its solution be

$$F(y, p, c) = 0, \qquad \ldots(3)$$

where c is the arbitrary constant.

Eliminating p between (1) and (3), we get the required solution of (1) in the form

$$\psi(x, y, c) = 0.$$

In case it is not easily practicable to eliminate p between (1) and (3), we may solve (1) and (3) to get x and y in terms of p and c in the form

$$x = f_1(p, c),\ y = f_2(p, c), \qquad \ldots(4)$$

which give us the required solution of (1) in the form of parametric equations, the parameter being p.

Special Case: **Equations that do not contain y:**

In this case the equation has the form $f(x, p) = 0$.

If it is solvable for p, it will give

$$p = \phi(x)$$

i.e., $dy/dx = \phi(x)$, which can be easily integrated.

If it is solvable for x, it will give $x = \psi(p)$, which can be solved by the method just explained in article 4.

Illustrative Examples

Example 10: *Solve* $y = 2px + y^2p^3$.

(Purvanchal 2007, 10; Lucknow 10; Avadh 07; Kumaun 07)

Solution: Solving for x, the given differential equation can be written as

$$2px = y - y^2p^3$$

or $$x = \frac{y}{2p} - \frac{y^2 p^2}{2}. \qquad \ldots(1)$$

Differentiating (1) w.r.t. y and writing $1/p$ for dx/dy, we obtain

$$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{2yp^2}{2} - \frac{y^2}{2}\cdot 2p\frac{dp}{dy}$$

or $$\frac{1}{2p} + yp^2 + \left(\frac{y}{2p^2} + py^2\right)\frac{dp}{dy} = 0$$

or $$p\left(\frac{1}{2p^2} + py\right) + y\left(\frac{1}{2p^2} + py\right)\frac{dp}{dy} = 0$$

or $$\left(\frac{1}{2p^2} + py\right)\left(p + y\frac{dp}{dy}\right) = 0\,.$$

∴ $$p + y\frac{dp}{dy} = 0 \quad \text{or} \quad \frac{1}{2p^2} + py = 0\,.$$

The equation $\frac{1}{2p^2} + py = 0$ will give us the singular solution of (1).

From $p + y\frac{dp}{dy} = 0$, we have

$$\frac{dp}{dy} = -\frac{p}{y} \quad \text{or} \quad (1/p)dp = -(1/y)\,dy.$$

Integrating, we get

$$\log p = -\log y + \log c \quad \text{or} \quad \log py = \log c$$

or $$py = c \quad \text{or} \quad p = c/y.$$

Substituting this value of p in the given differential equation, we get

$$y = 2x\cdot(c/y) + y^2(c^3/y^3) \quad \text{or} \quad y = 2cx/y + c^3/y$$

or $y^2 = 2cx + c^3$, which is the required solution.

Example 11: *Solve the differential equation*

$$p^3 - 4xyp + 8y^2 = 0\,.$$

(Garhwal 2008)

Solution: Solving the given differential equation for x, we get

$$4xyp = p^3 + 8y^2 \quad \text{or} \quad x = \frac{p^2}{4y} + \frac{2y}{p}. \qquad \ldots(1)$$

Differentiating (1) w.r.t. y and writing $1/p$ for dx/dy, we get

$$\frac{1}{p} = -\frac{p^2}{4y^2} + \frac{2p}{4y}\frac{dp}{dy} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}$$

or $$\frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right) = \frac{p^2}{4y^2} - \frac{1}{p}$$

or $$\frac{dp}{dy}\left(\frac{p}{2y}-\frac{2y}{p^2}\right)=\frac{p}{2y}\left(\frac{p}{2y}-\frac{2y}{p^2}\right)$$

or $$\frac{dp}{dy}=\frac{p}{2y},$$ cancelling the common factor on either side which corresponds to the singular solution of (1)

or $(2/p)\,dp=(1/y)\,dy$, separating the variables.

Integrating, we get

$$2\log p=\log y+\log c \qquad \text{or} \qquad \log p^2=\log cy$$

or $$p^2=cy \qquad \text{. } \qquad \ldots(2)$$

Now we shall eliminate p between the given differential equation and the equation (2). From the given differential equation, we have

$$8y^2=p\,(4xy-p^2)$$

or $8y^2=(cy)^{1/2}(4xy-cy)$, substituting for p from (2)

or $$8y^2=c^{1/2}y^{3/2}(4x-c) \qquad \text{or} \qquad 8y^{1/2}=c^{1/2}(4x-c)$$

or $$64y=c\,(4x-c)^2 \qquad \text{or} \qquad 64y=16c\,(x-\tfrac{1}{4}c)^2,$$

or $$y=\frac{1}{4}c\,(x-\frac{1}{4}c)^2 \qquad \text{or} \qquad y=c\,(x-c),$$ writing c for $\frac{1}{4}c$.

Hence the required solution is $y=c\,(x-c)^2$.

Example 12: *Solve* $ap^2+py-x=0$. **(Rohilkhand 2005; Meerut 13B)**

Solution: Solving the given differential equation for x, we get

$$x=py+ap^2. \qquad \ldots(1)$$

Differentiating (1) w.r.t. y and writing $1/p$ for dx/dy, we get

$$\frac{1}{p}=p+y\,\frac{dp}{dy}+2ap\,\frac{dp}{dy}$$

or $$\frac{1-p^2}{p}=y\,\frac{dp}{dy}+2ap\,\frac{dp}{dy}$$

or $$\frac{1-p^2}{p}\,\frac{dy}{dp}-y=2ap,$$ multiplying both sides by dy/dp

or $$\frac{dy}{dp}+\frac{p}{p^2-1}\,y=-\frac{2ap^2}{p^2-1}, \qquad \ldots(2)$$

which is a linear differential equation.

Here the I.F. $=e^{\int\{p/(p^2-1)\}dp}=e^{\frac{1}{2}\log(p^2-1)}=(p^2-1)^{1/2}$.

$\therefore$ the solution of (2) is

$$y\,(p^2-1)^{1/2}=\int\frac{-2ap^2}{p^2-1}(p^2-1)^{1/2}\,dp+c=-2a\int\frac{(p^2-1)+1}{\sqrt{(p^2-1)}}\,dp+c$$

$$= -2a \int \left[\sqrt{(p^2-1)} + \frac{1}{\sqrt{(p^2-1)}} \right] dp + c$$

$$= -2a \left[\frac{1}{2} p \sqrt{(p^2-1)} - \frac{1}{2} \cosh^{-1} p + \cosh^{-1} p \right] + c$$

$$= -ap \sqrt{(p^2-1)} - a \cosh^{-1} p + c$$

or $$y = \frac{c - a \cosh^{-1} p}{\sqrt{(p^2-1)}} - ap. \quad \ldots(3)$$

Substituting this value of y in (1), we get

$$x = p \left(\frac{c - a \cosh^{-1} p}{\sqrt{(p^2-1)}} - ap \right) + ap^2 \quad \text{or} \quad x = \frac{p(c - a \cosh^{-1} p)}{\sqrt{(p^2-1)}}. \quad \ldots(4)$$

The equations (3) and (4) constitute the parametric equations of the required solution.

Comprehensive Exercise 3

Solve the following differential equations :

1. $y^2 \log y = xyp + p^2$. **(Agra 2007; Purvanchal 08)**
2. $p^2 y + 2px - y = 0$. **(Meerut 2005; Purvanchal 06; Lucknow 08; Kumaun 10)**
3. $4(xp^2 + yp) = y^4$.
4. $(2x - b)p = y - ayp^2$.
5. $xp^3 = a + bp$.
6. $p = \tan\{x - p/(1+p^2)\}$.
7. $x + p/\sqrt{(1+p^2)} = a$.

Answers 3

1. $\log y = cx + c^2$
2. $y^2 - 2cx + c^2 = 0$
3. $y = 4c(xyc + 1)$
4. $ac^2 + (2x - b)c - y^2 = 0$
5. $x = (a/p^3) + (b/p^2), y = (3a/2p^2) + (2b/p) + c$
6. $x = p/(1+p^2) + \tan^{-1} p, y = c - 1/(1+p^2)$
7. $(x-a)^2 + (y+c)^2 = 1$

5 Clairaut's Equation

(Meerut 2004; Lucknow 08)

(a) *The differential equation*

$$y = px + f(p), \quad \ldots(1)$$

is known as Clairaut's equation. Here $f(p)$ is some function of p only. This differential equation is very important and the students should note its form carefully. To solve this differential equation, we shall obviously adopt the method explained in article 3. Thus differentiating (1) w.r.t. x and writing p for dy/dx, we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or $$\frac{dp}{dx}\{x + f(p)\} = 0 .$$

$\therefore$ $$\frac{dp}{dx} = 0 . \qquad \ldots(2)$$

or $$x + f'(p) = 0 . \qquad \ldots(3)$$

From (2), we have $p =$ constant $= c$, (say). ...(4)

Eliminating p between (1) and (4), we obtain

$$y = cx + f(c), \qquad \ldots(5)$$

which is the required general solution of (1).

If we eliminate p between (1) and (3), we get the **singular solution** of (1).

Remember: To obtain the general solution of a differential equation in Clairaut's form simply replace p by c.

(b) *To solve the differential equation*

$$y = xf_1(p) + f_2(p). \qquad \ldots(1)$$

This differential equation is not in Clairaut's form. However it can be solved by the method we adopted in solving Clairaut's equation. Thus differentiating (1) w.r.t. x and writing p for dy/dx, we get

$$p = f_1(p) + xf_1'(p)\frac{dp}{dx} + f_2'(p)\frac{dp}{dx}$$

or $$p - f_1(p) = xf_1'(p)\frac{dp}{dx} + f_2'(p)\frac{dp}{dx}$$

or $$[p - f_1(p)]\frac{dx}{dp} - x f_1'(p) = f_2'(p)$$

or $$\frac{dx}{dp} + \frac{f_1'(p)}{f_1(p) - p}\cdot x = \frac{f_2'(p)}{p - f_1(p)} \qquad \ldots(2)$$

which is a linear differential equation with x as the dependent variable and p as the independent variable.

Let the solution of (2) be

$$\phi(x, p, c) = 0 . \qquad \ldots(3)$$

Then eliminating p between (1) and (3), we get the required solution.

Illustrative Examples

Example 13: *Solve* $y = x(dy/dx) + (dy/dx)^2$. **(Meerut 2011; Kumaun 14; Kanpur 15)**

Solution: Denoting dy / dx by p, the given differential equation is $y = px + p^2$, which is in Clairaut's form. So replacing p by the arbitrary constant c, the required solution is

$$y = cx + c^2.$$

Example 14: *Solve* $p = \log (px - y)$. **(Gorakhpur 2007; Lucknow 09; Garhwal 09)**

Solution: The given differential equation can be written as $px - y = e^p$ or $y = px - e^p$, which is in Clairaut's form. So replacing p by the arbitrary constant c in the given differential equation, the required solution is $c = \log (cx - y)$.

Example 15: *Solve* $\sin px \cos y = \cos px \sin y + p$.

(Meerut 2003; Rohilkhand 06, 08, 10; Lucknow 07)

Solution: The given differential equation is

$$\sin px \cos y - \cos px \sin y = p$$

or $$\sin (px - y) = p \quad \text{or} \quad px - y = \sin^{-1} p$$

or $$y = px - \sin^{-1} p, \text{ which is in Clairaut's form.}$$

So changing p to the arbitrary constant c, the required solution is

$$y = cx - \sin^{-1} c.$$

Example 16: *Solve* $p^2 (x^2 - a^2) - 2pxy + y^2 - b^2 = 0$.

Solution: The given differential equation can be written as

$$p^2 x^2 - 2pxy + y^2 = p^2 a^2 + b^2$$

or $$(y - px)^2 = b^2 + a^2 p^2 \quad \text{or} \quad y = px \pm \sqrt{(b^2 + a^2 p^2)},$$

each of which is in Clairaut's form. Hence the required solution is

$$(y - cx)^2 = b^2 + a^2 c^2.$$

Example 17: *Solve* $y = 2px + p^n$.

Solution: The given differential equation is

$$y = 2px + p^n. \qquad \ldots(1)$$

Differentiating (1) w.r.t. x and writing p for dy/dx, we get

$$p = 2p + 2x (dp/dx) + np^{n-1} (dp/dx)$$

or $$p + 2x (dp/dx) = -np^{n-1} (dp/dx)$$

or $$p (dx/dp) + 2x = -np^{n-1}, \text{ multiplying both sides by } dx/dp$$

or $$\frac{dx}{dp} + \frac{2}{p} x = -np^{n-2}, \qquad \ldots(2)$$

which is a linear differential equation.

Hence the I.F. $= e^{\int (2/p)\, dp} = e^{2 \log p} = e^{\log p^2} = p^2$.

$\therefore$ the solution of (2) is

$$xp^2 = -\int n\, p^{n-2} p^2\, dp + c$$

or $$xp^2 = -n\int p^n\,dp + c = -n\,p^{n+1}/(n+1) + c$$

or $$x = cp^{-2} - \{n/(n+1)\}p^{n-1}. \qquad \ldots(3)$$

Substituting this value of x in (1), we get

$$y = 2p[cp^{-2} - \{n/(n-1)\}p^{n-1}] + p^n$$

or $$y = 2cp^{-1} + p^n - \frac{2n}{n+1}p^n = 2cp^{-1} - \frac{n-1}{n+1}p^n. \qquad \ldots(4)$$

The equations (3) and (4), which express x and y in terms of a parameter p, constitute the required solution.

6 Equations Reducible to Clairaut's Form

Some differential equations by suitable change of variables may be reduced to Clairaut's form.

Example 18: *Use the transformation* $x^2 = u$, $y^2 = v$ *to solve*

$$(px - y)(py + x) = h^2 p.$$

(Agra 2006; Lucknow 08, 10; Kanpur 15)

Solution: The given differential equation can be written as

$$p^2xy + p(x^2 - y^2 - h^2) - xy = 0. \qquad \ldots(1)$$

Putting $x^2 = u$ and $y^2 = v$, we have $2x\,dx = du$ and $2y\,dy = dv$.

$$\therefore \qquad \frac{y}{x}\frac{dy}{dx} = \frac{dv}{du} \quad \text{or} \quad \frac{dy}{dx} = \frac{x}{y}\frac{dv}{du} \quad \text{or} \quad p = \frac{x}{y}\frac{dv}{du}.$$

Putting this value of p in the differential equation (1), we get

$$xy\frac{x^2}{y^2}\left(\frac{dv}{du}\right)^2 + \frac{x}{y}\frac{dv}{du}(x^2 - y^2 - h^2) - xy = 0$$

or $$\frac{x}{y}\left[x^2\left(\frac{dv}{du}\right)^2 + (x^2 - y^2 - h^2)\frac{dv}{du} - y^2\right] = 0$$

or $$x^2(dv/du)^2 + (x^2 - y^2 - h^2)(dv/du) - y^2 = 0$$

or $$u(dv/du)^2 + (u - v - h^2)(dv/du) - v = 0,$$

[putting u for x^2 and v for y^2]

or $$uP^2 + (u - v - h^2)P - v = 0, \text{ where } P = dv/du$$

or $$uP(P+1) - v(P+1) - h^2 P = 0$$

or $$v(P+1) = uP(P+1) - h^2P$$

or $v = uP - h^2P/(P+1)$, which is in Clairaut's form $y = px + f(p)$.

$\therefore$ replacing P by the arbitrary constant c, we get the required solution as

$$v = uc - h^2c/(c+1) \qquad \text{or} \qquad y^2 = cx^2 - h^2c/(c+1).$$

Comprehensive Exercise 4

Solve the following differential equations :

1. $y = px + a/p$. **(Rohilkhand 2006, 07; Agra 05; Gorakhpur 08, 11; Purvanchal 06)**
2. $y = px + ap(1 - p)$. **(Agra 2006)**
3. $y = px + p - p^3$. **(Kanpur 2010)**
4. $(y - px)(p - 1) = p$. **(Agra 2005; Gorakhpur 06; Lucknow 09, 11)**
5. $(x - a)p^2 + (x - y)p - y = 0$. **(Agra 2007; Garhwal 07, 08)**
6. $p^2(x^2 - a^2) - 2pxy + y^2 + a^2 = 0$.
7. $y^2 + x^2(dy/dx)^2 - 2xy(dy/dx) = 4(dx/dy)^2$.
8. $\cos y \cos px + \sin y \sin px = p$
9. $9(y + xp \log p) = (2 + 3 \log p)p^3$.
10. $x^2(y - px) = yp^2$. **(Gorakhpur 2006, 09)**
    [**Hint.** Put $x^2 = u$ and $y^2 = v$]
11. $e^{3x}(p - 1) + p^3 e^{2y} = 0$. [**Hint.** Put $e^x = u$ and $e^y = v$.]

Answers 4

1. $y = cx + a/c$
2. $y = cx + ac(1 - c)$
3. $y = cx + c - c^3$
4. $(y - cx)(c - 1) = c$
5. $(x - a)c^2 + (x - y)c - y = 0$
6. $c^2(x^2 - a^2) - 2cxy + y^2 + a^2 = 0$
7. $(y - cx)^2 = 4/c^2$
8. $y = cx - \cos^{-1} c$
9. $x = \frac{1}{3}p^2 + cp^{-1}, y = -xp \log p + \frac{1}{9}(2 + 3 \log p)p^3$
10. $y^2 = cx^2 + c^2$
11. $e^y = ce^x + c^3$

7 Geometrical Meaning of a Differential Equation of the First Order

(Avadh 2010)

Let $$f(x, y, p) = 0, \quad \ldots(1)$$

be a differential equation of the first order, where $p = dy/dx$.

Suppose the general solution of (1) is

$$\phi(x, y, c) = 0, \quad \ldots(2)$$

where c is the arbitrary constant.

For each real value of c the equation (2) represents a curve. Thus (2) is the equation of a family of curves; c being the parameter for the family. Hence *every differential equation of the first order represents a family of curves*.

Let us take any point (x_1, y_1) in the plane. If we substitute the coordinates of this point in (1) and (2) and solve the resulting equations in p and c, the values of c so obtained are the values of the parameter for the curves of the family (2) which pass through (x_1, y_1) and the values of p are the slopes of the tangents to these curves at the point (x_1, y_1). Naturally the degree of c in (2) must be equal to the degree of p in (1).

8 Singular Solutions

(Gorakhpur 2009)

Sometimes a differential equation (for degree higher than one) possesses a solution which does not contain any arbitrary constant and which cannot be derived from the general solution of the differential equation by giving a particular value to the arbitrary constant. Such a solution is called a **singular solution** and it is generally not included in the general solution of the differential equation.

The singular solution of a differential equation is given by the envelope of family of curves represented by that differential equation.

Whenever the envelope of the family of curves

$$\phi(x, y, c) = 0, \qquad \dots(1)$$

represented by the general solution of the differential equation

$$f(x, y, p) = 0 \qquad \dots(2)$$

exists, the equation of the envelope is the singular solution of the differential equation (2).

Suppose that the family of curves (1) possesses an envelope. For any point $P(x, y)$ on the envelope, there exists a curve of the family (1), say $\phi(x, y, c) = 0$, which touches the envelope at (x, y). The values of $x, y, dy/dx$ for the curve at P satisfy the differential equation (2). But the values of $x, y, dy/dx$ at P for the envelope are the same as for the curve. Hence the values of $x, y, dy/dx$ at each point of the envelope satisfy the differential equation (2). Consequently the envelope of (1) is also a solution of (2). This solution does not contain any arbitrary constant and in general, cannot be obtained from (1) by giving any particular value to the arbitrary constant c. Hence this envelope is the singular solution of (2).

9 Determination of Singular Solution with the Help of c-Discriminant and p-Discriminant Relations

The discriminant: Let $F(c) = 0$ be an algebraic equation. The discriminant of this equation is the simplest function of the coefficients the vanishing of which represents the condition that the equation should have two equal roots. Thus the discriminant of the quadratic $Ac^2 + Bc + C = 0$ is $B^2 - 4AC$. The equation

$B^2 - 4AC = 0$ is called the *discriminant relation*.

Let the given differential equation be

$$f(x, y, p) = 0 . \qquad \ldots(1)$$

and let its general solution be

$$\phi(x, y, c) = 0 . \qquad \ldots(2)$$

The envelope of the family of curves (2) is *contained* in the locus obtained on eliminating c, between (2) and

$$\frac{\partial \phi(x, y, c)}{\partial c} = 0 . \qquad \ldots(3)$$

Let the c-eliminant between (2) and (3) be

$$\psi(x, y) = 0 . \qquad \ldots(4)$$

Since the equation (4) may sometimes represent other loci besides the envelope, therefore only that part of the locus (4) is the singular solution which also satisfies the given differential equation (1). Therefore if the equation (4) fails to satisfy (1), we should resolve it into others that are simpler. Then we should try for each part whether it satisfies the differential equation (1) or not. Only those parts will constitute the singular solution which satisfy (1).

From our knowledge of the theory of equations we know that the c-eliminant between (2) and (3) gives the condition for the equation (2) in c to have two equal roots. Therefore we shall call this c-eliminant the **c-discriminant relation**. The singular solution is contained in this c-discriminant relation. From the geometrical point of view the c-discriminant relation (4) is the locus of the points (x, y) such that two of the curves of the family (2) through (x, y) coincide. Now if two curves of the family coincide, their tangents must also coincide and so the corresponding values of p *i.e.*, (dy / dx) given by (1) must also coincide. The condition for the equation (1) in p to have two equal roots is obtained on eliminating p between (1) and

$$\frac{\partial f(x, y, p)}{\partial p} = 0 . \qquad \ldots(5)$$

The p-eliminant between (1) and (5) is called the **p-discriminant relation** and the envelope and hence the singular solution is also contained in it.

Note 1: From the above discussion it is clear that if *p occurs only in the first degree in the differential equation, there will be no singular solution*.

Note 2: A differential equation which possesses a singular solution is not considered completely solved until the singular solution also has been found. Therefore while solving a differential equation of the first order but higher degree we must find the general solution and also discuss the singular solution.

10 Working Rule for Finding the Singular Solution

Let the given differential equation be $f\ (x, y, p) = 0$ . ...(1)

Find the general solution of (1) and let it be $\phi\ (x, y, c) = 0$ . ...(2)

Find the c-**discriminant relation** by eliminating c between

$$\phi\ (x, y, c) = 0 \text{ and } \frac{\partial \phi\ (x, y, c)}{\partial c} = 0\ .$$

Also find the ***p*-discriminant relation** by eliminating p between

$$f\ (x, y, p) = 0 \quad \text{and } \frac{\partial f\ (x, y, p)}{\partial p} = 0\ .$$

The locus common to both the c-discriminant and the p-discriminant relations gives us the singular solution of (1) provided it satisfies (1).

However, we can find only one of these two discriminant relations and only that part of the locus contained in it is the singular solution which also satisfies the given differential equation (1).

11 The Singular Solution of Clairaut's Equation

We know that the general solution of the Clairaut's equation

$$y = px + f\ (p) \qquad \text{...(1)}$$

is $$y = cx + f\ (c) \qquad \text{...(2)}$$

Differentiating (2) partially w.r.t. c, we get

$$0 = x + f'\ (c)\ . \qquad \text{...(3)}$$

The singular solution, which is the envelope of (2), is obtained by eliminating c between (2) and (3).

Now differentiating (1) partially w.r.t. p, we get

$$0 = x + f'\ (p). \qquad \text{...(4)}$$

The equation (1) and (4) differ from the equations (2) and (3) only in having p instead of c. Therefore the c-eliminant between (2) and (3) and the p-eliminant between (1) and (4) are identical and either of them gives us the singular solution. **Hence the singular solution of the Clairaut's equation (1) is obtained by eliminating *p* between (1) and (4).**

Note: The equation (4) is the same as the equation (3) of article 5 (a).

Illustrative Examples

Example 19: *Find the general and singular solutions of*

$$y^2 - 2pxy + p^2\ (x^2 - 1) = m^2\ .$$

(Bundelkhand 2010; Kumaun 15)

Solution: The given differential equation can be written as

$$(y-px)^2 = p^2 + m^2$$

or $$y - px = \pm\sqrt{(p^2+m^2)} \quad \text{or} \quad y = px \pm \sqrt{(p^2+m^2)},$$

either of which is in Clairaut's form. Hence the general solution is obtained on replacing p by the arbitrary constant c in the given differential equation. Thus the general solution is

$$(y-cx)^2 = c^2 + m^2$$

or $$c^2(x^2-1) - 2xyc + y^2 - m^2 = 0. \qquad \ldots(1)$$

Now from the given differential equation and from the general solution (1) both the p and the c discriminant relations are

$$4x^2y^2 - 4(x^2-1)(y^2-m^2) = 0$$

or $$x^2m^2 + y^2 - m^2 = 0 \quad \text{or} \quad y^2 + m^2x^2 = m^2,$$

which is the singular solution.

Example 20: *Find the general and singular solution of the differential equation.*

$$(xp-y)^2 = p^2 - 1.$$ **(Bundelkhand 2008; Lucknow 06)**

Solution: From the given differential equation, we have

$$xp - y = \pm\sqrt{(p^2-1)} \quad \text{or} \quad y = px \pm \sqrt{(p^2-1)}$$

either of which is in Clairaut's form. Hence the general solution is obtained on replacing p by the arbitrary constant c in the given differential equation.

Thus the general solution is

$$(xc-y)^2 = c^2 - 1$$

or $$c^2(x^2-1) - 2xyc + y^2 + 1 = 0. \qquad \ldots(1)$$

The envelope of the family of curves (1) is the singular solution.

From (1) the c-discriminant relation is

$$4x^2y^2 - 4(x^2-1)(y^2+1) = 0$$

or $$-x^2 + y^2 + 1 = 0$$

or $$x^2 - y^2 = 1,$$

which is obviously also the p-discriminant relation. Since the c-discriminant relation contains only one locus, therefore it gives us the envelope of the family of curves (1). Hence the singular solution is $x^2 - y^2 = 1$.

Example 21: *Find the general and singular solution of*

$$9p^2(2-y)^2 = 4(3-y).$$

Solution: Solving the given differential equation for p, we have

$$p = \frac{dy}{dx} = \pm\frac{2}{3}\frac{\sqrt{(3-y)}}{2-y}$$

or $$dx = \pm \frac{3}{2} \frac{2-y}{\sqrt{(3-y)}} dy \text{ , separating the variables.}$$

Integrating, we have

$$x + c = \pm \frac{3}{2} \int \frac{2-y}{\sqrt{(3-y)}} dy = \pm \frac{3}{2} \int \frac{2-(3-t^2)}{t} (-2t)\, dt \text{ ,}$$

putting $3 - y = t^2$ so that $-dy = 2t\, dt$

$$= \pm 3 \int (t^2 - 1)\, dt = \pm 3 \left(\frac{1}{3} t^3 - t\right) = \pm t\, (t^2 - 3)$$

$$= \pm \sqrt{(3-y)}\, (-y), \qquad [\because 3 - y = t^2]$$

$$\therefore \qquad (x+c)^2 = y^2\, (3-y), \qquad \ldots(1)$$

which is the general solution of the given differential equation.

Differentiating (1) partially w.r.t. c, we get

$$2\,(x+c) = 0 \quad i.e., \quad x + c = 0 \text{ .} \qquad \ldots(2)$$

Eliminating c between (1) and (2), we get the c-discriminant relation as

$$y^2\, (3-y) = 0 \text{ .}$$

Now $y = 0$ gives $dy/dx = p = 0$. Putting $y = 0$ and $p = 0$ in the given differential equation, we see that these values of y and p do not satisfy it. Therefore $y = 0$ is not a solution.

Again $3 - y = 0$ gives $y = 3$ and so $dy/dx = p = 0$. Putting $y = 3$ and $p = 0$ in the given differential equation, we see that these values of y and p satisfy it. Therefore $y = 3$ is the singular solution.

Example 22: *Reduce the differential equation* $(px - y)(x - py) = 2p$ *to Clairaut's form by substituting* $x^2 = u$ *and* $y^2 = v$ *and find its complete primitive and the singular solution.*

Solution: We put $u = x^2$ and $v = y^2$, so that

$$du/dx = 2x \quad \text{and} \quad dv/dx = 2y\,(dy/dx) = 2py.$$

$$\therefore \qquad \frac{dv}{du} = \frac{dv/dx}{du/dx} = \frac{2py}{2x} = \frac{py}{x}$$

or $\quad P = py/x$, where $P = dv/du$

or $\quad p = xP/y$.

Substituting this value of P in the given differential equation, we get

$$\left(\frac{x^2 P}{x} - y\right)(x - x\,P) = 2x\frac{P}{y} \qquad \text{or} \qquad x\,(x^2\; P - y^2)\,(1-P) = 2x\; P$$

or $\quad (x^2 P - y^2)\,(1 - P) = 2P \qquad$ or $\quad (u\; P - v)\,(1-P) = 2P$

or $\quad u\; P - v = 2P/(1 - P)$

or $\quad v = uP - 2P/(1-P)$,

which is in Clairaut's form.

Hence its general solution is $v = uc - 2c/(1-c)$

or $\quad y^2 = x^2 c - 2c/(1-c) \quad$ or $\quad y^2 (1-c) = x^2 c (1-c) - 2c$

or $\quad c^2 x^2 - c(x^2 + y^2 - 2) + y^2 = 0 . \qquad \ldots(1)$

Now (1) is a quadratic in c. So the c-discriminant relation is

$$(x^2 + y^2 - 2)^2 - 4x^2 y^2 = 0$$

or $\quad (x^2 - 2xy + y^2 - 2)(x^2 + 2xy + y^2 - 2) = 0$

or $\quad \{(x-y)^2 - 2\}\{(x+y)^2 - 2\} = 0$

or $\quad (x - y - \sqrt{2})(x - y + \sqrt{2})(x + y - \sqrt{2})(x + y + \sqrt{2}) = 0 .$

All the four equations $x - y - \sqrt{2} = 0$, $x - y + \sqrt{2} = 0$, $x + y - \sqrt{2} = 0$ and $x + y + \sqrt{2} = 0$ satisfy the given differential equation and are therefore the singular solutions.

Comprehensive Exercise 5

1. Find the complete primitive and singular solution of $y = px + \sqrt{(b^2 + a^2 p^2)}$. Interpret your results geometrically. **(Avadh 2010)**
2. Find the general and singular solutions of $y = px + a/p$.
3. Solve and examine for singular solution the differential equation $x^3 p^2 + x^2 yp + a^3 = 0$. **(Gorakhpur 2007)**
4. Find the general and singular solution of $27y - 8p^3 = 0$.
5. Solve and examine for singular solution the equation $xp^2 - (x-a)^2 = 0$.
6. Find the general and singular solution of
   (a) $(y - px)^2 + a^2 p = 0$. **(b)** $3xy = 2px^2 - 2p^2$.
   (c) $p^2 + y^2 = 1$.
7. Examine $y^2 (1 + p^2) = r^2$ for singular solution. **(Bundelkhand 2009)**
8. Reduce the equation $x^2 p^2 + py(2x + y) + y^2 = 0$, where $p = (dy/dx)$ to Clairaut's form by putting $u = y$ and $v = xy$ and find its complete primitive and also its singular solution. **(Gorakhpur 2010)**
9. Solve the differential equation $(px^2 + y^2)(px + y) = (p+1)^2$ by reducing it to Clairaut's form and find its singular solution. **(Purvanchal 2009)**

10. Reduce the differential equation $y = 2px + y^2 p^3$ to Clairaut's form by putting $y^2 = v$ and hence find its general and singular solutions.

11. Solve and test for singular solutions $p^3 - 4pxy + 8y^2 = 0$.

Answers 5

1. $y = cx + \sqrt{(b^2 + a^2c^2)}; x^2/a^2 + y^2/b^2 = 1$
2. $y = cx + a/c; y^2 = 4ax$
3. $c^2 + cxy + a^3x = 0$; $x = 0$, $xy^2 - 4a^3 = 0$
4. $y^2 = (x + c)^3$, $y = 0$
5. $9(y + c)^2 = 4x(x - 3a)^2$, $x = 0$
6. **(a)** $(y - cx)^2 + a^2c = 0$; $xy = a^2/4$ **(b)** $(3y + 2c)^2 = 4cx^3$; $x^3 - 6y = 0$

   (c) $y = \cos(x + c)$; $y = \pm 1$
7. $y^2 + (x + c)^2 = r^2$; $y = \pm r$
8. $xy = cy + c^2$, $y = 0$, $y + 4x = 0$
9. $c^2(x + y) - cxy - 1 = 0; x^2y^2 + 4(x + y) = 0$
10. $y^2 = cx + c^3/8$; $27y^4 + 32x^3 = 0$
11. $y = c(x - c)^2$; $y = 0$, $y = \frac{4}{27}x^3$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Solution of the differential equation $p^2 - 8p + 15 = 0$ is

   (a) $p = 5$, $p = 3$ (b) $(y - 5x - c)(y - 3x - c) = 0$

   (c) $(y + 5x)(y + 3x + c) = 0$ (d) none of these

   (Kumaun 2006, 08)

2. Solution of the equation $y^2 \log y = xyp + p^2$ is

   (a) $\log y = cx + x^2$ (b) $\log y = cx^2 + e^x$

   (c) $\log x = cy + y^2$ (d) none of these.

3. Solution of the equation $y = px + p^2$ is

   (a) $y = e^x + c$ (b) $y = cx + c^2$

   (c) $y = \log cx$ (d) $x = e^y + c$.

4. Which of the following equations is Clairaut's equation ?

(a) $x = py + f(p)$ (b) $y = px + f(c)$

(c) $y = px + f(p)$ (d) $y = x^2 + c$

5. Solution of the equation $p^2 - 5p + 6 = 0$ is

(a) $y = 2x + c$ (b) $y = 3x + c$

(c) $(y - 2x - c)(y - 3x - c) = 0$ (d) none of these

(Garhwal 2007; Kanpur 16)

6. Solution of the equation $y = px + \frac{a}{p}$ is

(a) $y = cx$ (b) $y = cx + \frac{a}{c}$

(c) $y + cx = a$ (d) $cy = x + \frac{a}{c}$

(Garhwal 2008; Kumaun 14; Kanpur 16)

7. Solution of the equation $(x - a)p^2 + (x - y)p - y = 0$ is

(a) $y = cx - \frac{ac^2}{c+1}$ (b) $y = cy + \frac{ac^2}{x+1}$

(c) $y = cx - \frac{ac^2}{y+1}$ (d) $x = cy - \frac{ac^2}{c+1}$ **(Garhwal 2009)**

8. Solution of equation $y = px + \log p$ is

(a) $y = e^x + c$ (b) $y = cx + \log c$

(c) $y = \log cx$ (d) $x = e^y + c$ **(Garhwal 2011)**

9. The solution of the equation $y + px = x^4 p^2$ is

(a) $xy = -c + c^2 x$ (b) $x^2 = c - c^2 x$

(c) $y^2 = cx + c$ (d) $xy^2 = cx - c$ **(Garhwal 2010)**

10. The solution of $y = px + f(p)$ is

(a) $y = cx + f(c)$ (b) $y + cx = f(c)$

(c) $y - cx = f(c)$ (d) $y = f(c) - cx$

(Garhwal 2005, 06, 14)

11. Solution of the differential equation $y = px + e^p$ is

(a) $y = -ax + e^p$ (b) $y = ax + e^a$

(c) $y = ax + e^{-a}$ (d) None of these

(Kumaun 2006; Garhwal 15)

12. General solution of differential equation $\sin(y - px) = p$ is

(a) $y = cx - \sin^{-1} c$ (b) $y = cx - \cos^{-1} c$

(c) $y = cx + \sin^{-1} c$ (d) $y = cx^2 + \sin^{-1} c$ **(Kanpur 2016)**

13. Singular solution of $y = px + \frac{a}{p}$ represents

(a) ellipse (b) circle

(c) parabola (d) none of these **(Kanpur 2016)**

14. General solution of differential equation $p = \log(px - y)$ is

(a) $y = cx - e^c$ (b) $y = cx + e^c$

(c) $y = cx^2 + e^{2x}$ (d) $y = cx^2 - e^{2x}$ **(Kanpur 2016)**

15. General solution of differential equation $\left(\dfrac{dy}{dx}\right)^2 - ax^3 = 0$ is

(a) $25(y - c) = 4ax^5$ (b) $25(y + c) = 4ax^5$

(c) $25(y + c)^2 = 4ax^5$ (d) $25(y - c)^2 = 4ax^5$ **(Kanpur 2016)**

16. General solution of the differential equation $xp^2 - yp + a = 0$ is

(a) $xc^2 - yc + a = 0$ (b) $xc^2 + yc + a = 0$

(c) $\dfrac{x}{c^2} - \dfrac{y}{c} + a = 0$ (d) none of these

17. General solution of differential equation $y = x\dfrac{dy}{dx} + e^{(dy/dx)}$ is

(a) $y = cx + ce$ (b) $y = cx + e^c$

(c) $y = cx + e^{1/c}$ (d) none of these

18. Let $p = \dfrac{dy}{dx}$, then the solution of $y = px + \sqrt{1 - p^2}$ is

(a) $y = cx$ (b) $y = cx + \sqrt{1 - c^2}$

(c) $y = x^2 + c$ (d) $y = c_1 + c_2 x$ **(Kanpur 2016)**

19. General solution of the differential equation $y = px + p - p^3$ is

(a) $y = cx + c - c^3$ (b) $y = x + c - c^3$

(c) $y = x - c^2 + c^3$ (d) none of these

Fill in The Blanks

Fill in the blanks "……" so that the following statements are complete and correct.

1. We usually denote $\dfrac{dy}{dx}$ by the letter …… .

2. The differential equation $y = px + f(p)$, is known as …… .

(Meerut 2003; Avadh 05; Kumaun 10)

3. $p^n + A_1 p^{n-1} + A_2 p^{n-2} + \ldots + A_{n-1} p + A_n = 0$,

where $A_1, A_2, \ldots, A_n$ are some functions of x and y, is a differential equation of …… order and …… degree.

4. Solution of the equation $p^2 - 7p + 12 = 0$ is …… .

5. Solution of the equation $y = px + ap(1 - p)$ is …… .

6. Solution of the differential equation $y = px + e^p$ is ....

7. Solution of the equation $y = x\left(\dfrac{dy}{dx}\right) + \left(\dfrac{dy}{dx}\right)^3$ is .... **(Kumaun 2007, 09)**

8. Solution of the equation $\cos y \cos px + \sin y \sin px = p$ is ....

9. Solution of the equation $(x - a)\, p^2 + (x + y)\, p - y = 0$ is ....

True or False

Write 'T' for true and 'F' for false statement.

1. Every differential equation of the first order represents a family of curves.

2. The singular solution of a differential equation is given by the envelope of the family of curves represented by that differential equation.

3. Only those differential equations possess a singular solution in which p (or dy/dx) occurs only in the first degree.

Answers

Multiple Choice Questions

1. (b) 2. (a) 3. (b) 4. (c) 5. (c)
6. (b) 7. (a) 8. (b) 9. (a) 10. (a)
11. (b) 12. (c) 13. (c) 14. (a) 15. (c)
16. (a) 17. (b) 18. (b) 19. (a)

Fill in the Blank(s)

1. p
2. Clairaut's equation
3. first, nth
4. $(y - 3x - c)(y - 4x - c) = 0$
5. $y = cx + ac\,(1 - c)$
6. $y = cx + e^c$
7. $y = cx + c^3$
8. $y = cx - \cos^{-1} c$
9. $(x - a)\, c^2 + (x - y)\, c - y = 0$

True or False

1. T 2. T 3. F

Chapter

3

Orthogonal Trajectories

1 Trajectory

(Lucknow 2005; Gorakhpur 06; Bundelkhand 08; Meerut 13, 13B)

Definition:. *A trajectory of a given system of curves is defined to be a curve which cuts all the members of the family according to a given law*. Here we propose to find the equation of the trajectories (a family of curves) each member of which cuts each member of a given family of curves at a constant angle. If the angle is a right angle, the trajectories are called **orthogonal trajectories**, when it is other than a right angle they are called to be **oblique trajectories**.

2 Trajectories

Cartesian Co-ordinates: *To find the trajectories which cut every member of a given family of curves at a constant angle*.

Let the equation of the given family of curves be

$$f\ (x, y, c) = 0\ , \qquad \ldots(1)$$

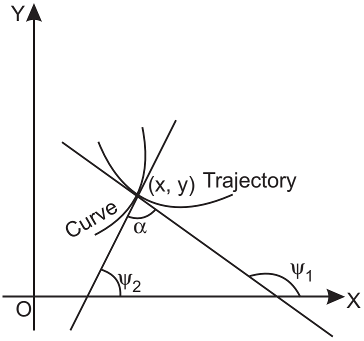
c being the arbitrary parameter. Let the required trajectories cut the given curves (1) at a constant angle α.

Differentiating (1) with respect to x, we have

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0\,. \qquad \ldots(2)$$

Eliminating c between (1) and (2), we get the differential equation of the given family of curves (1). Let it be

$$\phi\left(x, y, \frac{dy}{dx}\right)=0\,. \qquad \ldots(3)$$



Let (x, y) be the coordinates of a point of intersection of a member of the given curves (1) and a member of the required trajectories. If the tangent to the trajectory at this point makes an angle ψ_1 with x-axis and if the tangent to the given curve at this point makes an angle ψ_2 with x-axis, we have

$$dy/dx \text{ given by } (3)=\tan\psi_2=\tan(\psi_1-\alpha)=\frac{\tan\psi_1-\tan\alpha}{1+\tan\psi_1\tan\alpha}\,.$$

Putting this value of (dy/dx) in (3), we get

$$\phi\left(x, y, \frac{\tan\psi_1-\tan\alpha}{1+\tan\psi_1\tan\alpha}\right)=0\,. \qquad \ldots(4)$$

Now $dy\,/\,dx$ for the trajectory is $\tan\psi_1$. So replacing $\tan\psi_1$ in the equation (4) by $dy\,/\,dx$, we get the differential equation of the required trajectories as

$$\phi\left(x, y, \frac{dy/dx-\tan\alpha}{1+(dy/dx)\tan\alpha}\right)=0\,.$$

Solving this differential equation, we shall obtain the equation of the required trajectories.

Orthogonal trajectories: If α is a right angle, we have

$$\tan\psi_1\times\tan\psi_2=-1\,.$$

$$\therefore \qquad \frac{dy}{dx} \text{ given by } (3)=\tan\psi_2=-\frac{1}{\tan\psi_1}\,.$$

Putting this value of dy/dx in (3) of the last article, we get

$$\phi\left(x, y, -\frac{1}{\tan\psi_1}\right)=0\,. \qquad \ldots(5)$$

Now (dy/dx) for the trajectory is $\tan\psi_1$. So replacing $\tan\psi_1$ in the equation (5) by dy/dx, we get differential equation of the orthogonal trajectories of (1) as

$$\phi\left(x, y, -\frac{dx}{dy}\right)=0\,.$$

Solving this differential equation, we shall get the equation of the orthogonal trajectories of the given family (1).

Remember: *To obtain the differential equation of the orthogonal trajectories, we have to* ***write*** ***– dx/dy for dy/dx*** *in the differential equation of the original family of curves.*

3 Orthogonal Trajectories

Polar Co-ordinates: Let the equation of the given family of curves be

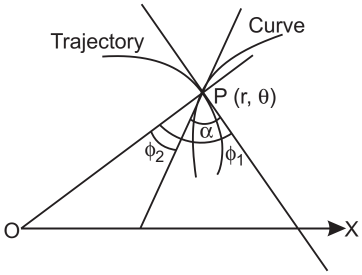
$$f\ (r, \theta, c) = 0\ , \qquad \dots(1)$$

c being the parameter *i.e.*, c can take any real value. Differentiating (1) w.r.t. θ and then eliminating c, let the differential equation of the given family (1) be

$$\phi\left(r, \theta, \frac{dr}{d\theta}\right) = 0\ . \qquad \dots(2)$$

Now if at a point of intersection $P\ (r, \theta)$, the tangents to the trajectory and the given curve make angles ϕ_1 and ϕ_2 with the common radius vector OP of that point, then

$$\tan 90^\circ = \tan(\phi_1 - \phi_2) = \frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \tan \phi_2}$$



and therefore $\tan \phi_1 \tan \phi_2 = -1$.

Now $r \dfrac{d\theta}{dr}$ given by (2)

$$= \tan \phi_2 = -\frac{1}{\tan \phi_1}\cdot$$

$\therefore$ $\qquad dr/d\theta$ given by (2) $= -r \tan \phi_1$.

$\therefore$ $\qquad$ putting this value of $dr/d\theta$ in (2), we have

$$\phi\ (r, \theta, -r \tan \phi_1) = 0$$

But $\tan \phi_1 = r \dfrac{d\theta}{dr}$ for the trajectory.

$\therefore$ $\qquad$ the differential equation of the required orthogonal trajectories is

$$\phi\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0\ .$$

Solving this we shall get the required equation of the orthogonal trajectories.

Remember: *The differential equation of the orthogonal trajectories is obtained from the differential equation (in polar co-ordinates) of a given family of curves by writing*

$$-r^2 \frac{d\theta}{dr} \quad \text{for} \quad \frac{dr}{d\theta} \quad \text{or} \quad -r\frac{d\theta}{dr} \quad \text{for} \quad \frac{1}{r}\frac{dr}{d\theta}$$

or $\qquad -\dfrac{1}{r}\dfrac{dr}{d\theta} \quad$ for $\quad r\dfrac{d\theta}{dr}\cdot$

Illustrative Examples

Example 1: *Find the orthogonal trajectories of the family of curves* $y = ax^n$.

(Kumaun 2008, 12; Meerut 11; Kanpur 15)

Solution: The equation of the given family of curves is

$$y = ax^n, \ a \text{ being the parameter} \qquad \dots.(1)$$

Differentiating (1) with respect to x, we get

$$dy/dx = a\,.\,nx^{n-1}. \qquad \ldots(2)$$

Dividing (2) by (1) to eliminate a between (1) and (2), we have

$$\frac{dy/dx}{y} = \frac{anx^{n-1}}{ax^n} = \frac{n}{x} \quad \text{or} \quad x\,(dy/dx) = ny, \qquad \ldots(3)$$

which is the differential equation of the given family of curves. Now to obtain the differential equation of the orthogonal trajectories, write $-\,dx/dy$ for dy/dx in (3).

Hence the orthogonal trajectories are given by the differential equation

$$x\,(-\,dx/dy) = ny. \qquad \ldots(4)$$

Separating the variables, we have $-\,x\,dx = ny\,dy$.

Integrating, we have $-\frac{1}{2}x^2 = n\frac{1}{2}y^2 + c$, where c is constant of integration.

Thus $x^2 + ny^2 = 2c$ or $x^2 + ny^2 = c^2$ is the required family of orthogonal trajectories.

Example 2: *Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$, where 'a' is the variable parameter.* **(Meerut 2013B; Kanpur 09)**

Solution: Differentiating the given equation $y^2 = 4ax$, $\ldots(1)$

with respect to x, we get

$$2y\,(dy/dx) = 4a\,. \qquad \ldots(2)$$

Eliminating a between (1) and (2), we get

$$2x\,(dy/dx) = y, \qquad \ldots(3)$$

which is the differential equation of the given family of parabolas.

So to obtain the differential equation of the orthogonal trajectories, write $-\,dx/dy$ for dy/dx in (3). Thus the differential equation of the required family of orthogonal trajectories is

$$2x\,(-dx/dy) = y \quad \text{or} \quad 2x\,dx = -\,y\,dy.$$

Integrating, we get

$$x^2 = -\frac{1}{2}y^2 + c^2 \quad \text{or} \quad 2x^2 + y^2 = k^2,$$

which is the required family of orthogonal trajectories.

Example 3: *Find the orthogonal trajectories of the family of rectangular hyperbolas $xy = c^2$.*

(Meerut 2003; Rohilkhand 06, 07; Kanpur 07, 12; Bundelkhand 09; Avadh 10, 11)

Solution: The equation of the given family of curves is

$$xy = c^2,\ c \text{ being the parameter}. \qquad \ldots(1)$$

Differentiating (1) w.r.t. x, we get

$$x\,(dy/dx) + y = 0\,, \qquad \ldots(2)$$

which is independent of the parameter c. Hence the equation (2) is the differential equation of the given family of curves.

So to obtain the differential equation of the orthogonal trajectories, write $-\,dx/dy$ for dy/dx in (2).

Thus the differential equation of the required family of orthogonal trajectories is

$$x\,(-\,dx/dy)+y=0$$

or $\quad x\,dx=y\,dy$, in which the variables are separated.

$\therefore\quad$ Integrating, $\dfrac{1}{2}x^2=\dfrac{1}{2}y^2+c$, where c is constant of integration

or $\quad x^2-y^2=2c=k^2$ (say) is the required family of orthogonal trajectories.

Example 4: *Find the orthogonal trajectories of the family of parabolas $y^2=4a\,(x+a)$, where 'a' is the parameter.* **(Meerut 2006; Purvanchal 06; Lucknow 06; Avadh 07, 09)**

Solution: The equation of the given family of parabolas is

$$y^2=4a\,(x+a)\,, \qquad \ldots(1)$$

a being the parameter.

Differentiating (1) w.r.t. x, we get

$$2y\,(dy/dx)=4a. \qquad \ldots(2)$$

Now eliminating a between (1) and (2), we get

$$y^2=2y\,(dy/dx)\,[x+\frac{1}{2}\,y\,(dy/dx)]$$

or

$$y=2x\,(dy/dx)+y(dy/dx)^2, \qquad \ldots(3)$$

which is the differential equation of the given family of parabolas. So to obtain the differential equation of the orthogonal trajectories, write $-\,dx/dy$ for dy/dx in (3).

Thus the differential equation of the required family of orthogonal trajectories is

$$y=2x\,(-\,dx/dy)+y\,(-\,dx/dy)^2$$

or

$$y\,(dy/dx)^2+2x\,(dy/dx)=y. \qquad \ldots(4)$$

Now we observe that the differential equation (4) of the orthogonal trajectories is the same as the differential equation (3) of the given family of parabolas. Therefore the given family of parabolas (1) is self-orthogonal *i.e.*, the orthogonal trajectories of the system belong to the system itself. Hence the equation of the orthogonal trajectories of (1) is $y^2=4c\,(x+c)$, c being the parameter.

Example 5: *Find the orthogonal trajectory of the family of circles $x^2+y^2=2ax$, a being the parameter.* **(Meerut 2005, 10)**

Solution: Differentiating

$$x^2+y^2=2ax \qquad \ldots(1)$$

with respect ot x, we get

$$2x+2y\,(dy/dx)-2a=0$$

or

$$x+y\,(dy/dx)-a=0. \qquad \ldots(2)$$

Eliminating 'a' between (1) and (2), we get

$$x^2 + y^2 = 2x\,\{x + y\,(dy/dx)\}$$

or $$x^2 + 2xy\,(dy/dx) - y^2 = 0, \qquad \ldots(3)$$

which is the differential equation of the given family of circles. So to obtain the differential equation of the orthogonal trajectories, write $-\,dx/dy$ for dy/dx in (3).

Thus the differential equation of the required family of orthogonal trajectories is

$$x^2 + 2xy\,(-\,dx/dy) - y^2 = 0$$

or $$dx/dy = (x^2 - y^2)/2xy. \qquad \ldots(4)$$

This is a homogeneous differential equation. To solve it putting $x = vy$, so that

$dx/dy = v + y\,(dv/dy)$, in (4), we get

$$v + y\,(dv/dy) = y^2\,(v^2 - 1)/(2vy^2)$$

or $$y\,\frac{dv}{dy} = \frac{v^2 - 1}{2v} - v = -\,\frac{1 + v^2}{2v}$$

or $$\frac{2v\,dv}{1 + v^2} = -\,\frac{dy}{y}.$$

Integrating, we have

$$\log\,(1 + v^2) = -\log\,y + \log\,b$$

or $$1 + v^2 = b/y$$

or $$1 + (x^2/y^2) = b/y, \text{ since } v = x/y.$$

Hence $x^2 + y^2 = by$ is the equation of the required orthogonal trajectories.

Example 6: *Find the orthogonal trajectories of the family of curves*

$$\{x^2/(a^2 + \lambda)\} + \{y^2/(b^2 + \lambda)\} = 1,$$

where λ *is the parameter.* **(Meerut 2007B, 09B; Lucknow 10; Rohilkhand 10; Gorakhpur 11; Purvanchal 07, 09, 11)**

Solution: Differentiating the given equation with respect to x, we get

$$2x/(a^2 + \lambda) + \{2y/(b^2 + \lambda)\} \cdot (dy/dx) = 0$$

or $$x\,(b^2 + \lambda) + y\,(dy/dx)\,(a^2 + \lambda) = 0$$

or $$\lambda\,\{x + y\,(dy/dx)\} = -\,\{b^2 x + a^2 y\,(dy/dx)\}.$$

$\therefore$ $$\lambda = -\,\{b^2 x + a^2 y\,(dy/dx)\} \div \{x + y\,(dy/dx)\}.$$

Thus $$a^2 + \lambda = (a^2 - b^2)\,x/\,\{x + y\,(dy/dx)\}$$

and $$b^2 + \lambda = -\,(a^2 - b^2)\,y\,(dy/dx) \div \{x + y\,(dy/dx)\}.$$

Substituting these values of $(a^2 + \lambda)$ and $(b^2 + \lambda)$ in the given equation, we get the differential equation of the given family of curves as

$$\frac{x^2\,\{x + y\,(dy/dx)\}}{(a^2 - b^2)\,x} - \frac{y^2\{x + y(dy/dx)\}}{(a^2 - b^2)\,y\,(dy/dx)} = 1$$

or $$x^2 - y^2 + xy\left(\frac{dy}{dx} - \frac{1}{dy/dx}\right) = a^2 - b^2. \qquad \ldots(1)$$

Hence putting $-\,dx/dy$ for dy/dx in (1), the differential equation of the orthogonal trajectories is

$$x^2 - y^2 + xy\left(-\frac{dx}{dy} + \frac{dy}{dx}\right) = a^2 - b^2$$

or $$x^2 - y^2 + xy\left(\frac{dy}{dx} - \frac{1}{dy/dx}\right) = a^2 - b^2,$$

which is the same as (1). Therefore solving it we shall get

$$x^2 / (a^2 + \mu) + y^2 (b^2 + \mu) = 1,$$

μ being the parameter of the orthogonal trajectories. In other words the system of given confocal conics $x^2 / (a^2 + \lambda) + y^2 / (b^2 + \lambda) = 1$ is self-orthogonal.

Example 7: *Find the orthogonal trajectories of the cardioids* $r = a\,(1 - \cos\theta)$, *a being the parameter.* **(Meerut 2006B; Gorakhpur 05; Lucknow 11; Kashi 11)**

Solution: The given family of curves is

$$r = a\,(1 - \cos\theta), \qquad \ldots(1)$$

the parameter being a. Differentiating (1) w.r.t. θ, we get

$$dr/d\theta = a \sin\theta\,. \qquad \ldots(2)$$

Now we shall eliminate a between (1) and (2). So dividing (1) by (2), we get

$$r\,(d\theta / dr) = (1 - \cos\theta) / \sin\theta, \qquad \ldots(3)$$

which is the differential equation of the given family of curves (1).

To obtain the differential equation of the orthogonal trajectories, write $-\,(1/r)\,(dr/d\theta)$ for $r\,(d\theta/dr)$ in (3). Thus we get

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{1 - \cos\theta}{\sin\theta} = \frac{2\sin^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \tan\frac{1}{2}\theta,$$

which is the differential equation of the orthogonal trajectories. To solve it, separating the variables, we get

$$\frac{dr}{r} = -\tan\frac{1}{2}\theta\,d\theta = \frac{-\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\,d\theta = 2\cdot\frac{-\frac{1}{2}\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\,d\theta\,\cdot$$

Integrating, we have

$$\log r = 2\log\cos\frac{1}{2}\theta + \log c$$

or $$\log\,(r/c) = \log\cos^2\frac{1}{2}\theta$$

or $$r/c=\cos^2\frac{1}{2}\theta=\frac{1}{2}(1+\cos\theta)$$

or $$r=\frac{1}{2}c\,(1+\cos\theta)$$

or $$r=k\,(1+\cos\theta),$$

which is the required family of orthogonal trajectories.

Example 8: *Find the orthogonal trajectories of the system of curves* $r^n=a^n\cos n\theta$, *a being the parameter.* **(Meerut 2007, 10B; Bundelkhand 05; Gorakhpur 06; Lucknow 08; Kanpur 08)**

Solution: The given family of curves is $r^n=a^n\cos n\theta$, ...(1)

the parameter being a. Taking logarithm of both sides of (1), we get

$$n\log r=n\log a+\log\cos n\theta\,.$$

Now differentiating both sides w.r.t. θ, we get

$$\frac{n}{r}\frac{dr}{d\theta}=0+\frac{-n\sin n\theta}{\cos n\theta}$$

or $$\frac{1}{r}\frac{dr}{d\theta}=-\tan n\theta \qquad \text{...(2)}$$

which is the differential equation of the family of curves (1).

Now putting $-r\dfrac{d\theta}{dr}$ for $\dfrac{1}{r}\dfrac{dr}{d\theta}$ in (2), the differential equation of the orthogonal trajectories is

$$-r\frac{d\theta}{dr}=-\tan n\theta$$

or $$\frac{dr}{r}=\cot n\theta\,d\theta\,.$$

Integrating, we get $\log r=(1/n)\log\sin n\theta+\log c$

or $$n\log(r/c)=\log\sin n\theta$$

or $$\log(r/c)^n=\log\sin n\theta$$

or $$r^n=c^n\sin n\theta,$$

which is the required family of orthogonal trajectories, the parameter being c.

Comprehensive Exercise 1

1. Find the equation of the family of curves orthogonal to the family $y=ax^3$.
(Avadh 2005; Rohilkhand 09; Kanpur 11)
2. Find the orthogonal trajectories of the semi-cubical parabolas $ay^2=x^3$, where a is the variable parameter. **(Gorakhpur 2007; Kashi 13; Kumaun 15)**

3. Find the orthogonal trajectories of the family of circles $x^2 + y^2 = a^2$, where 'a' is the parameter. **(Bundelkhand 2008; Kanpur 10; Kumaun 10, 13)**
4. Find the orthogonal trajectories of the family of curves $x^{2/3} + y^{2/3} = a^{2/3}$, where a is parameter.
5. Find the equation of the family of curves that is orthogonal to $ax^2 + y^2 = 1$.
6. Find the orthogonal trajectories of the system of curves $(dy/dx)^2 = a/x$. **(Bundelkhand 2007, 10; Agra 08; Kumaun 11; Purvanchal 14)**
7. Find the differential equation of the family of curves given by the equation $x^2 - y^2 + 2\lambda xy = 1$, where λ is a parameter.

   Obtain the differential equation of its orthogonal trajectories and solve it.
8. Show that the orthogonal trajectories of the family of conics $y^2 - x^2 + 4xy - 2cx = 0$ consist of a family of cubics, with the common asymptote $x + y = 0$. **(Meerut 2009)**
9. Find the orthogonal trajectories of the family of circles $x^2 + y^2 + 2fy + 1 = 0$. where f is the parameter. **(Meerut 2004; Gorakhpur 08, 11)**
10. Find the orthogonal trajectories of the family of curves
$$\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1, \lambda \text{ being parameter.}$$
(Lucknow 2005; Meerut 05B; Purvanchal 14)
11. Find the orthogonal trajectories of the family of coaxial circles $x^2 + y^2 + 2gx + c = 0$, where g is a parameter and c is a constant. **(Agra 2007)**
12. Find the orthogonal trajectories of

    (i) $r\theta = a$, (ii) $r = a\theta$. **(Rohilkhand 2008)**
13. Find the orthogonal trajectories of the family of cardioids $r = a(1 + \cos\theta)$. **(Rohilkhand 2006; Lucknow 06; Purvanchal 08, 10; Gorakhpur 10)**
14. Find the orthogonal trajectories of $r = e^{a\theta}$. **(Kumaun 2009)**
15. Find the orthogonal trajectories of $r^n \sin n\theta = a^n$. **(Meerut 2001, 04B)**

Answers 1

1. $x^2 + 3y^2 = c^2$
2. $x^2 + \frac{3}{2}y^2 = c^2$
3. $x = cy$
4. $x^{4/3} - y^{4/3} = c^{4/3}$
5. $y^2 = ce^{(x^2 + y^2)}$
6. $(x^{3/2} + c^{3/2})^2 = \frac{9}{4}ay^2$
7. $\frac{1}{2}(x^2 + y^2)^2 + y^2 - x^2 = c^2$
8. $y^3 + 3x^2y + 4x^3 = k^3$
9. $y^2 + x^2 - 1 = cx$
10. $x^2 + y^2 - 2a^2 \log x = c$

11. $x^2 + y^2 + 2fy - c = 0$, f being parameter.

12. **(i)** $r^2 = c^2 \cdot e^{\theta^2}$ **(ii)** $r = ce^{-\frac{1}{2}\theta^2}$

13. $r = c\,(1 - \cos\theta)$ **14.** $\theta^2 + (\log r)^2 = c^2$

15. $r^n \cos n\theta = c^n$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. To find the orthogonal trajectories in polar form, we replace $\frac{dr}{d\theta}$ by

(a) $-\frac{dr}{d\theta}$ (b) $\frac{d\theta}{dr}$

(c) $-r^2 \frac{d\theta}{dr}$ (d) $r^2 \frac{d\theta}{dr}$ **(Kumaun 2010)**

2. Family of orthogonal trajectories for the curve $xy = c$ is

(a) $y^2 - x^2 = 2c$ (b) $y^2 = 2c + x$

(c) $y = x^2 + c$ (d) $e^y - e^x = c$ **(Kumaun 2014)**

3. Orthogonal trajectories of the system of curves $\left(\frac{dy}{dx}\right)^2 = \frac{a}{x}$ is

(a) $9a\,(y - c)^2 = 4x^3$ (b) $9a\,(y + c)^2 = 4x^3$

(c) $9a^2\,(y + 2c)^2 = 4x^2$ (d) $9a\,(y - 2c)^2 = 4x^2$

(Kanpur 2016)

4. Orthogonal trajectories of cardioids $r = a\,(1 - \cos\theta)$, a being a parameter, is

(a) $r = \frac{c}{2}(1 + \cos\theta)$ (b) $r = \frac{c}{2}(1 - \cos\theta)$

(c) $r^2 = \frac{c}{2}(1 - \cos\theta)$ (d) $r = \frac{c}{2}(a - \sin^2\theta)$ **(Kanpur 2016)**

5. Orthogonal trajectories of the family of parabolas $y^2 = 4ax$, a being a parameter, is

(a) $x^2 + y^2 = k^2$ (b) $x^2 - y^2 = k^2$

(c) $2x^2 + y^2 = k^2$ (d) $x^2 + 2y^2 = k^2$

6. Equation of the family of curves orthogonal to the family $y = ax^3$ is

(a) $3x^2 + y^2 = c^2$ (b) $x^2 + 3y^2 = c^2$

(c) $3x^2 - y^2 = c^2$ (d) $x^2 - 3y^2 = c^2$

7. Orthogonal trajectories of the semi-cubical parabolas $ay^2 = x^3$, a is the variable parameter, is

(a) $x^2 - \frac{3}{2} y^2 = c^2$ (b) $x^2 + \frac{3}{2} y^2 = c^2$

(c) $\frac{3}{2} x^2 - y^2 = c^2$ (d) $\frac{3}{2} x^2 + y^2 = c^2$

8. Orthogonal trajectories of $r\theta = a$ is

(a) $r^2 = c^2 e^{\theta}$ (b) $r^2 = c^2 e^{-\theta}$

(c) $r^2 = c^2 e^{\theta^2}$ (d) $r = c^2 e^{\theta^2}$

9. Orthogonal trajectories of $r = a\theta$ is

(a) $r = ce^{-\theta^2/2}$ (b) $r = ce^{\theta^2/2}$

(c) $r = ce^{-\theta/2}$ (d) $r = ce^{\theta^3/2}$

10. Orthogonal trajectories of $r = e^{a\theta}$ is

(a) $\theta^2 + \log r = c^2$ (b) $\theta^2 + (\log r)^2 = c^2$

(c) $\theta^2 - \log r = c^2$ (d) $\theta^2 - (\log r)^2 = c^2$

11. Orthogonal trajectories of the family of curves $y = ax^n$ is

(a) $x^2 + ny^2 = c^2$ (b) $nx^2 + y^2 = c^2$

(c) $x^2 - ny^2 = c^2$ (d) $nx^2 - y^2 = c^2$

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. The differential equation of the orthogonal trajectories is obtained from the differential equation of the given family of curves by writing …… for dy/dx.
2. The differential equation of the orthogonal trajectories is obtained from the differential equation (in polar coordinates) of the given family of curves by writing …… for $\frac{dr}{d\theta}$.
3. Orthogonal trajectories of the system of curves $r^n = a^n \cos n\theta$, a being the parameter, is ……

True or False

Write 'T' for true and 'F' for false statement.

1. A family of curves is said to be self orthogonal if the differential equation of the orthogonal trajectories is the same as the differential equation of the given family of curves.
2. Family of curves $\{x^2/(a^2 + \lambda)\} + \{y^2/(b^2 + \lambda)\} = 1$, where λ is the parameter, is self orthogonal.

Answers

Multiple Choice Questions

1. (c) **2.** (a) **3.** (b) **4.** (a) **5.** (c)
6. (b) **7.** (b) **8.** (c) **9.** (a) **10.** (b)
11. (a)

Fill in the Blank(s)

1. $-\dfrac{dx}{dy}$ **2.** $-r^2\dfrac{d\theta}{dr}$ **3.** $r^n = c^n \sin n\theta$

True or False

1. T **2.** T

□

Chapter 4

Linear Differential Equations with Constant Coefficients

1 Definitions

A linear differential equation is an equation in which the dependent variable and its derivatives appear only in the first degree.

A linear differential equation of order n of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = Q\,, \qquad \ldots(1)$$

where $a_1, a_2, \ldots, a_{n-1}, a_n$ are constants and Q is any function of x is called a **linear differential equation with constant coefficients.**

For convenience, the operators $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \ldots, \frac{d^n}{dx^n}$ are also denoted by D, $D^2, D^3, \ldots, D^n$ respectively.

Thus the equation (1) can also be written as

$$D^n y + a_1 D^{n-1} y + \ldots + a_{n-1} Dy + a_n y = Q$$

or

$$[D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n]\, y = Q\,. \qquad \ldots(2)$$

If $y = f(x)$ is the general solution of

$$[D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n]\ y = 0, \qquad \ldots(3)$$

and $y = \phi(x)$ is any particular solution of the equation (2) not containing any arbitrary constant, then

$$y = f(x) + \phi(x),$$

is the general solution of (2).

Thus the method of solving a linear equation is divided into two parts :

First, we find the general solution of the equation (3).

It is called the *complementary function* (C.F.). It must contain as many arbitrary constants as is the order of the given differential equation.

Next, we find a solution of (2) which does not contain an arbitrary constant. This is called the *particular integral* (P.I.).

If we add (C.F.) and (P.I.), we get the general solution of (2). Thus the general solution of (2) is

$$y = \textbf{C. F.} + \textbf{P. I.}$$ **(Gorakhpur 2005)**

2 Determination of Complementary Function (C.F.)

Consider a linear nth order differential equation with constant coefficients of the form $f(D)\ y = 0$ *i.e.*,

$$[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n]\ y = 0. \qquad \ldots(1)$$

This is equivalent to

$$[(D - m_1)(D - m_2) \ldots (D - m_n)]\ y = 0. \qquad \ldots(2)$$

The solution of any one of the equations

$$(D - m_1)\ y = 0, (D - m_2)\ y = 0, \ldots, (D - m_n)\ y = 0 \qquad \ldots(3)$$

is also a solution of (2) and we know that the general solution of $(D - m_1)\ y = 0$ is

$$y = Ae^{m_1 x}.$$

Hence we can assume that a solution of the equation (2) is of the form $y = e^{mx}$. Then, substituting e^{mx} for y in (1), so that $Dy = me^{mx}$, $D^2 y = m^2 e^{mx}$, ..., $D^n y = m^n e^{mx}$, we get

$$e^{mx}(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n) = 0$$

or $$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n = 0, \text{ because } e^{mx} \neq 0.$$

Hence e^{mx} will be a solution of (1) if m has the value obtained from the equation $m^n + a_1 m^{n-1} + \ldots + a_n = 0$. ...(4)

The equation (4) is called the **auxiliary equation (A.E.)** and is obtained by putting $D = m$ in $f(D) = 0$.

It will give in general n roots, say, $m_1, m_2, m_3, \ldots, m_n$.

Case I: If all the roots of the Auxiliary equation (A.E.) are distinct:

If the roots $m_1, m_2, m_3, \ldots, m_n$ are all distinct, then $e^{m_1 x}, e^{m_2 x}, \ldots, e^{m_n x}$ are all distinct and linearly independent. So the general solution of (1) in this case is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}. \qquad \ldots(5)$$

Case II: Auxiliary equation having equal roots :

If two roots are equal say $m_1 = m_2$, then the solution (5) becomes

$$y = c_1 e^{m_1 x} + c_2 e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

or $$y = (c_1 + c_2)\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}.$$

Now $(c_1 + c_2)$ can be replaced by single constant say c.

Therefore this solution has only $(n-1)$ arbitrary constants and so it is not the general solution.

To obtain the general solution, consider the differential equation $(D - m_1)^2\, y = 0$ in which the two roots are equal.

This can be written as $(D - m_1)\,[(D - m_1)\, y] = 0$. $\qquad \ldots(6)$

Now putting $(D - m_1)\, y = v$, we get $(D - m_1)\, v = 0$,

or $\quad dv/dx = m_1 v \quad$ or $\quad dv/v = m_1\, dx$, (variables being separated).

$\therefore$ integrating,

$$\log v = m_1 x + \log c_1 \quad \text{or} \quad \log (v/c_1) = m_1 x \quad \text{or} \quad v = c_1 e^{m_1 x}.$$

Thus $(D - m_1)\, y = v = c_1 e^{m_1 x}$

and putting it in (6), we get $(D - m_1)\, y = c_1 e^{m_1 x}$

or $$Dy - m_1 y = c_1 e^{m_1 x} \quad \text{or} \quad \frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}. \qquad \left[\because D \equiv \frac{d}{dx}\right]$$

This is a linear equation of the first order. Hence the I. F. $= e^{\int -m_1\, dx} = e^{-m_1 x}$.

$\therefore$ The solution of this equation is

$$y \cdot e^{-m_1 x} = \int c_1 e^{m_1 x} \cdot e^{-m_1 x}\, dx + c_2$$

$$= c_1 \int dx + c_2 = c_1 x + c_2$$

or $$y = (c_1 x + c_2)\, e^{m_1 x} \text{ or } y = (c_2 + c_1 x)\, e^{m_1 x}.$$

Hence the general solution of $f(D)\, y = 0$ in this case is

$$y = (c_1 + c_2 x)\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}.$$

Similarly if three roots of the auxiliary equation are equal say, $m_1 = m_2 = m_3$, the general solution of $f(D)\, y = 0$ will be

$$y = (c_1 + c_2 x + c_3 x^2)\, e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

and so on.

Case III: Auxiliary equation having complex roots: **(Meerut 2003)**

Let the two roots of the auxiliary equation be complex, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, (where $i = \sqrt{-1}$).

The solution corresponding to these two roots will be

$$y = c_1 e^{(\alpha + i\beta)\,x} + c_2 e^{(\alpha - i\beta)\,x} = c_1 e^{\alpha x} \cdot e^{i\beta x} + c_2 e^{\alpha x} \cdot e^{-i\beta x}$$

$$= c_1 e^{\alpha x} (\cos \beta x + i \sin \beta x) + c_2 e^{\alpha x} (\cos \beta x - i \sin \beta x)$$

$$= (c_1 + c_2)\, e^{\alpha x} \cos \beta x + i\,(c_1 - c_2)\, e^{\alpha x} \sin \beta x$$

or $y = e^{\alpha x} [A_1 \cos \beta x + A_2 \sin \beta x]$, where $A_1 = c_1 + c_2$ and $A_2 = i\,(c_1 - c_2)$

or $y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$, chaning the constants.

If the imaginary roots are repeated, say $\alpha + i\beta$ and $\alpha - i\beta$ occur twice then the solution will be

$$y = e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4\, x) \sin \beta x],$$

and so on.

Note 1: The expression $e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$ can also be written as

$$c_1 e^{\alpha x} \sin (\beta x + c_2) \qquad \text{or} \qquad c_1 e^{\alpha x} \cos (\beta x + c_2).$$

Note 2: If a pair of the roots of the auxiliary equation are irrational *i.e.*, they are $\alpha \pm \sqrt{\beta}$, where β is positive, then the corresponding term in the C.F. will be

$$e^{\alpha x} (c_1 \cosh \sqrt{\beta} x + c_2 \sinh \sqrt{\beta} x),$$

or $c_1 e^{\alpha x} \sinh (\sqrt{\beta} x + c_2)$ or $c_1 e^{\alpha x} \cosh (\sqrt{\beta} x + c_2)$.

If these irrational roots are repeated twice, then the corresponding portion of the solution will be

$$e^{\alpha x} \{(c_1 + c_2 x) \cosh \sqrt{\beta} x + (c_3 + c_4 x) \sinh \sqrt{\beta} x\}.$$

Illustrative Examples

Example 1: *Solve* $\dfrac{d^2 y}{dx^2} - 7 \dfrac{dy}{dx} + 12 y = 0$.

Solution: The given differential equation is $(D^2 - 7D + 12)\, y = 0$.

$\therefore$ the auxiliary equation is $m^2 - 7m + 12 = 0$

or $(m - 3)(m - 4) = 0$. $\therefore$ $m = 3, 4$.

Hence the solution is

$$y = c_1 e^{3x} + c_2 e^{4x}.$$

Example 2: Solve $(D^3 + 6D^2 + 11D + 6)\, y = 0$. **(Meerut 2010)**

Solution: The auxiliary equation is

$$m^3 + 6m^2 + 11m + 6 = 0$$

or $(m + 1)(m^2 + 5m + 6) = 0$

or $(m + 1)(m + 2)(m + 3) = 0$.

$\therefore \qquad m = -1, -2, -3 .$

Hence the solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}.$$

Example 3: *Solve* $\dfrac{d^2 x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 0$ *, given that when* $t = 0, x = 0$ *and* $dx/dt = 0$ *.*

(Gorakhpur 2008)

Solution: The auxiliary equation is $m^2 - 3m + 2 = 0$

or $\qquad (m-1)(m-2) = 0 \qquad$ or $\qquad m = 1, 2 .$

Hence the solution is $x = c_1 e^t + c_2 e^{2t}$, ...(1)

where c_1 and c_2 are arbitrary constants.

Now $x = 0$ when $t = 0$;

$\therefore \qquad 0 = c_1 + c_2 .$ ...(2)

Also $dx/dt = c_1 e^t + 2c_2 e^{2t}$, and $dx/dt = 0$ when $t = 0$.

$\therefore \qquad 0 = c_1 + 2c_2 .$ ...(3)

Solving (2) and (3), we get $c_1 = 0, c_2 = 0$.

Now putting the values of c_1 and c_2 in (1), we get the required solution as $x = 0$.

Example 4: *Solve* $(D^3 - 3D + 2)\, y = 0$.

Solution: The auxiliary equation is $m^3 - 3m + 2 = 0$

or $\qquad (m-1)(m^2 + m - 2) = 0 \qquad$ or $\qquad (m-1)\{(m-1)(m+2)\} = 0 .$

$\therefore \qquad m = 1, 1, -2.$

Hence the solution is

$$y = (c_1 + c_2 x)\, e^x + c_3 e^{-2x}.$$

Example 5: *Solve* $\dfrac{d^3 y}{dx^3} - 8y = 0$.

(Meerut 2010B)

Solution: The auxiliary equation is $m^3 - 8 = 0$

or $\qquad (m-2)(m^2 + 2m + 4) = 0$ *i.e.,* $\quad m - 2 = 0$ and $\quad m^2 + 2m + 4 = 0.$

$\therefore \qquad m = 2$ and $m = -1 \pm i\sqrt{3}$.

Hence the solution is

$$y = e^{-x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x) + c_3 e^{2x}.$$

Example 6: *Solve* $\dfrac{d^4 y}{dx^4} - 2\dfrac{d^3 y}{dx^3} - 2\dfrac{dy}{dx} - y = 0$.

Solution: The auxiliary equation is

$$m^4 - 2m^3 - 2m - 1 = 0$$

or $\qquad (m^4 - 1) - 2m(m^2 + 1) = 0$

or $\qquad (m^2 + 1)(m^2 - 1) - 2m(m^2 + 1) = 0$

or $(m^2+1)(m^2-2m-1)=0$ *i.e.,* $m^2=-1$ or $m^2-2m-1=0$.

$\therefore$ $m=0\pm i, 1\pm\sqrt{2}$.

Hence the solution is

$$y=e^{0x}(c_1\cos x+c_2\sin x)+e^x(c_3\cosh\sqrt{2}x+c_4\sinh\sqrt{2}\,x)$$

or $$y=c_1\cos x+c_2\sin x+e^x\{c_3\cosh\sqrt{2}x+c_4\sinh\sqrt{2}x\}.$$

Example 7: *Solve* $(D^4+k^4)\,y=0$. **(Meerut 2003, 04B, 05B, 06)**

Solution: The auxiliary equation is $m^4+k^4=0$

or $(m^2+k^2)^2-2k^2m^2=0$

or $(m^2+k^2)^2-(\sqrt{2}km)^2=0$

or $(m^2+k^2-\sqrt{2}\,km)(m^2+k^2+\sqrt{2}\,km)=0$

or $m^2-\sqrt{2}\,km+k^2=0$ and $m^2+\sqrt{2}\,km+k^2=0$

or $$m=\frac{\sqrt{2}k\pm\sqrt{(2k^2-4k^2)}}{2} \text{ and } m=\frac{-\sqrt{2}k\pm\sqrt{(2k^2-4k^2)}}{2}$$

or $$m=\frac{k}{\sqrt{2}}\pm i\frac{k}{\sqrt{2}} \text{ and } -\frac{k}{\sqrt{2}}\pm i\frac{k}{\sqrt{2}}.$$

Hence the solution is

$$y=e^{kx/\sqrt{2}}\{c_1\cos(kx/\sqrt{2})+c_2\sin(kx/\sqrt{2})\}$$
$$+e^{-kx/\sqrt{2}}\{c_3\cos(kx/\sqrt{2})+c_4\sin(kx/\sqrt{2})\}.$$

Comprehensive Exercise 1

Solve the following differential equations :

1. $\dfrac{d^2y}{dx^2}+(a+b)\dfrac{dy}{dx}+aby=0.$

2. $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}-4y=0.$

3. $\dfrac{d^3y}{dx^3}+6\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}-10y=0.$

4. $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+4y=0.$

5. $(D^3-4D^2+5D-2)\,y=0.$

6. $\dfrac{d^4y}{dx^4}+2\dfrac{d^3y}{dx^3}-3\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+4y=0.$ **(Garhwal 2009)**

7. (i) $(D^4+1)\,y=0.$ **(Agra 2006)** (ii) $\dfrac{d^4y}{dx^4}-k^4y=0.$

8. $(D^4+8D^2+16)\,y=0.$

9. $(D^2\pm\mu^2)\,y=0.$ **(Avadh 2005, 14)**

10. $(D^3-2D^2-4D+8)\,y=0.$ **(Garhwal 2010)**

11. Solve $(d^2 y/dx^2) + y = 0$, given $y = 2$ for $x = 0$ and $y = -2$ for $x = \frac{1}{2}\pi$.

Answers 1

1. $y = c_1 e^{-ax} + c_2 e^{-bx}$

2. $y = c_1 e^{-x} + c_2 e^{4x}$

3. $y = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{-5x}$

4. $y = (c_1 + c_2 x) e^{2x}$

5. $y = (c_1 + c_2 x) e^{x} + c_3 e^{2x}$

6. $y = (c_1 + c_2 x) e^{x} + (c_3 + c_4 x) e^{-2x}$

7. (i) $y = e^{x/\sqrt{2}} \{c_1 \cos (x/\sqrt{2}) + c_2 \sin (x/\sqrt{2})\}$
$+ e^{-x/\sqrt{2}} \{c_3 \cos (x/\sqrt{2}) + c_4 \sin (x/\sqrt{2})\}$.

(ii) $y = c_1 e^{kx} + c_2 e^{-kx} + c_3 \cos kx + c_4 \sin kx$

8. $y = (c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x$

9. $y = c_1 \cos \mu x + c_2 \sin \mu x$ and $y = c_3 e^{\mu x} + c_4 e^{-\mu x}$

10. $y = (c_1 + c_2 x) e^{2x} + c_3 e^{-2x}$

11. $y = 2\sqrt{2} \cos \left(x + \frac{1}{4}\pi \right)$

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3 The Particular Integral (P.I.)

As already shown in article 1 the complete solution of

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n)\, y = Q \qquad \ldots(1)$$

or $\qquad F(D)\, y = Q$ is $y = \text{C.F.} + \text{P.I.}$,

where the C.F. consists of the general solution of the differential equation

$$F(D)\, y = 0.$$

In article 2 we have discussed different methods of finding the complementary function by taking the differential equation as $F(D)\, y = 0$. Methods of finding the particular integral will be discussed now.

The particular integral of the differential equation $F(D)\, y = Q$ is $\dfrac{1}{F(D)} Q$. It is obviously a function of x which when operated by $F(D)$ gives Q.

Now as $F(D) \dfrac{1}{F(D)} Q = Q$, therefore $\dfrac{1}{F(D)}$ can be regarded as the inverse operator of $F(D)$. Similarly D and $1/D$ are inverse operations. If D stands for differentiation then $1/D$ will stand for integration.

Hence the particular integral of the equation

$$F(D)\,y = Q \text{ will be } \frac{1}{F(D)}\,Q$$

because it satisfies the given equation.

4 Particular Integral in Some Special Cases

Case I: To find P.I. when Q is of the form e^{ax}, where a is any constant and $F(a) \neq 0$.

By simple differentiation we know that $D(e^{ax}) = ae^{ax}$;

$$D^2(e^{ax}) = a^2e^{ax};\ D^3(e^{ax}) = a^3e^{ax},\ldots,\ D^n(e^{ax}) = a^ne^{ax}.$$

It suggests that $F(D)\,e^{ax} = F(a)\,e^{ax}$. ...(1)

Let $F(a) \neq 0$.

Operating on both sides of (1) with $1/F(D)$, we have

$$\frac{1}{F(D)}\,F(D)\,e^{ax} = \frac{1}{F(D)}\{F(a)\,e^{ax}\}$$

or $$e^{ax} = F(a)\cdot\frac{1}{F(D)}\,e^{ax}, \text{ because } F(a) \text{ is a constant}$$

or $$\frac{1}{F(a)}\,e^{ax} = \frac{1}{F(D)}\,e^{ax}, \text{ because } F(a) \neq 0$$

$\therefore$ $$\text{P.I.} = \frac{1}{F(D)}\,e^{ax} = \frac{1}{F(a)}\,e^{ax}, \textbf{ provided } \boldsymbol{F(a) \neq 0}.$$

Working Rule : If P.I. $= \{1/F(D)\}\,e^{ax}$, then put a for D in $F(D)$ and we get the P.I., provided $F(a) \neq 0$.

Illustrative Examples

Examples on case I:

Example 8: *Solve* $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^{5x}$.

(Kumaun 2006; Avadh 08; Garhwal 10B; Purvanchal 11; Kanpur 12)

Solution: The given equation can be written as

$$(D^2 - 3D + 2)\,y = e^{5x}, \text{ where } d/dx \equiv D.$$

Here $$F(D) = D^2 - 3D + 2 \text{ and } Q = e^{5x}.$$

The auxiliary equation is

$$m^2 - 3m + 2 = 0 \quad\text{or}\quad (m-1)(m-2) = 0.$$

$\therefore$ $$m = 1, 2.$$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{2x}$, the roots of the A.E. being distinct.

and
$$\text{P.I.} = \frac{1}{F(D)} e^{ax} = \frac{1}{D^2 - 3D + 2} e^{5x} = \frac{e^{5x}}{5^2 - 3 \cdot 5 + 2} = \frac{e^{5x}}{12}.$$

[We have put 5 for D in $F(D)$, because here $a = 5$]

Hence the complete solution is $y =$ (C.F.) + (P.I.)

or
$$y = c_1 e^x + c_2 e^{2x} + \frac{1}{12} e^{5x}.$$

Example 9: *Solve* $\dfrac{d^2 y}{dx^2} + 31 \dfrac{dy}{dx} + 240 y = 272 e^{-x}$. **(Meerut 2010B, 11)**

Solution: The auxiliary equation is

$$m^2 + 31m + 240 = 0 \quad \text{or} \quad (m + 15)(m + 16) = 0 ;$$

$\therefore$ $m = -15, -16$.

$\therefore$ C.F. $= c_1 e^{-15x} + c_2 e^{-16x}$.

and
$$\text{P.I.} = \frac{1}{D^2 + 31D + 240} (272 e^{-x}) = 272 \cdot \frac{1}{D^2 + 31D + 240} e^{-x}$$
$$= 272 \cdot \frac{1}{(-1)^2 + 31(-1) + 240} e^{-x} = \frac{272}{210} e^{-x} = \frac{136}{105} e^{-x}.$$

Hence the general solution is $y =$ (C.F.) + (P.I.)

i.e.,
$$y = c_1 e^{-15x} + c_2 e^{-16x} + \frac{136}{105} e^{-x}.$$

Example 10: *Obtain the complete solution of the differential equation*

$$\frac{d^2 y}{dx^2} - 7 \frac{dy}{dx} + 6y = e^{2x},$$

and determine the constant so that $y = 0$ *when* $x = 0$.

Solution: The given equation is $[D^2 - 7D + 6]\, y = e^{2x}$.

$\therefore$ auxiliary equation is $m^2 - 7m + 6 = 0$

or $(m - 1)(m - 6) = 0$ or $m = 1, 6$.

$\therefore$ C.F. $= c_1 e^x + c_2 e^{6x}$.

and
$$\text{P.I.} = \frac{1}{D^2 - 7D + 6} e^{2x} = \frac{1}{2^2 - 7 \cdot 2 + 6} e^{2x} = -\frac{e^{2x}}{4}.$$

$\therefore$ the general solution is $y =$ (C.F.) + (P.I.)

i.e.,
$$y = c_1 e^x + c_2 e^{6x} - \frac{1}{4} e^{2x}. \qquad \ldots(1)$$

Now when $y = 0, x = 0$;

$\therefore$ $0 = c_1 + c_2 - \dfrac{1}{4}$, from (1)

so that $c_2 = \left(\dfrac{1}{4} - c_1\right)$.

Hence from (1), we have

$$y = c_1 e^x + \left(\frac{1}{4} - c_1\right) e^{6x} - \frac{1}{4} e^{2x} = c_1 (e^x - e^{6x}) + \frac{1}{4} (e^{6x} - e^{2x})$$

or $y = c_1 (e^x - e^{6x}) + \frac{1}{4} e^{2x} (e^{4x} - 1)$ is the required solution.

Comprehensive Exercise 2

Solve the following differential equations :

1. $\dfrac{d^2 y}{dx^2} - 2k \dfrac{dy}{dx} + k^2 y = e^x.$
2. $\dfrac{d^2 y}{dx^2} + 2 \dfrac{dy}{dx} + y = 2e^{2x}.$ **(Bundelkhand 2001)**
3. $\dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} + y = e^x.$
4. $[D^2 + D + 1]\, y = e^{-x}.$
5. $\dfrac{d^3 y}{dx^3} + 6 \dfrac{d^2 y}{dx^2} + 11 \dfrac{dy}{dx} + 6y = e^{2x}.$
6. $(D^3 + 1)\, y = (e^x + 1)^2.$

Answers 2

1. $y = (c_1 + c_2 x) e^{kx} + e^x / (1 - k)^2$
2. $y = (c_1 + c_2 x) e^{-x} + \dfrac{2}{9} e^{2x}$
3. $y = e^{-x/2} \{c_1 \cos (x \frac{1}{2} \sqrt{3}) + c_2 \sin (x \frac{1}{2} \sqrt{3})\} + \frac{1}{3} e^x$
4. $y = c_1 e^{-x/2} \cos \{x (\sqrt{3} / 2) + c_2\} + e^{-x}$
5. $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x} + \dfrac{1}{60} e^{2x}$
6. $y = \dfrac{1}{9} e^{2x} + e^x + 1$

Case II: To find P.I. when Q is of the form sin ax or cos ax and $F(-a^2) \neq 0$.

By simple differentiation we know that

$$D (\sin ax) = a \cos ax \,;\; D^2 (\sin ax) = - a^2 \sin ax \,;$$

$$D^3 (\sin ax) = - a^3 \cos ax \,;\; D^4 (\sin ax) = (- a^2)^2 \sin ax,$$

$$\ldots\ldots, (D^2)^n \sin ax = (- a^2)^n \sin ax.$$

It suggests that

$$F (D^2) \sin ax = F (- a^2) \sin ax. \qquad \ldots(1)$$

Let $F(-a^2) \neq 0.$

Now operating both sides of (1) with $1/F(D^2)$, we get

$$\frac{1}{F(D^2)} F(D^2) \sin ax = \frac{1}{F(D^2)} \{F(-a^2) \sin ax\}$$

or $\sin ax = F(-a^2) \cdot \frac{1}{F(D^2)} \sin ax.$

Thus $\boldsymbol{\frac{1}{F(D^2)} \sin ax = \frac{1}{F(-a^2)} \sin ax,}$ provided $F(-a^2) \neq 0$.

Similarly $\boldsymbol{\frac{1}{F(D^2)} \cos ax = \frac{1}{F(-a^2)} \cos ax,}$ provided $F(-a^2) \neq 0$.

Working Rule: If P.I. $= \{1/F(D)\} \sin ax$ or $\cos ax$, put $-a^2$ for $D^2, -a^2 D$ for D^3, $(-a^2)^2$ *i.e.*, a^4 for $D^4, a^4 D$ for $D^5, -a^6$ for D^6 etc. in $F(D)$ and calculate the P.I.

Note: Linear factors in D of the form $(pD \pm q)$ appearing in the denominator are removed by first multiplying the Nr. and Dr. by the conjugate factors $(pD \mp q)$ and then putting $-a^2$ for D^2 in the denominator. The operation left in the numerator can be easily worked out because the operator D stands for differentiation with respect to x.

Illustrative Examples

Examples on Case II:

Example 11: *Solve* $\frac{d^2 y}{dx^2} - \frac{dy}{dx} - 2y = \sin 2x.$ **(Meerut 2008, 10)**

Solution: The given equation is $[D^2 - D - 2]\, y = \sin 2x.$

$\therefore$ auxiliary equation is $m^2 - m - 2 = 0$

or $(m+1)(m-2) = 0$; $\therefore$ $m = -1, 2$.

$\therefore$ $\text{C.F.} = c_1 e^{-x} + c_2 e^{2x}.$

And $$\text{P.I.} = \frac{1}{D^2 - D - 2} \sin 2x = \frac{1}{-4 - D - 2} \sin 2x,$$

putting -2^2 *i.e.*, -4 for D^2

$$= -\frac{1}{D+6} \sin 2x = -\frac{(D-6)}{(D+6)(D-6)} \sin 2x = -\frac{(D-6)}{D^2 - 36} \sin 2x$$

$$= -\frac{(D-6)}{-4-36} \sin 2x, \text{ putting } -4 \text{ for } D^2$$

$$= \frac{1}{40} (D-6) \sin 2x = \frac{1}{40} \{D(\sin 2x) - 6 \sin 2x\}$$

$$= \frac{1}{40}(2\cos 2x - 6\sin 2x) = \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x .$$

$\therefore$ the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $$y = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x .$$

Example 12: *Solve* $(D^2 - 5D + 6)\, y = \sin 3x$. **(Kanpur 2015)**

Solution: Here the auxiliary equation is $m^2 - 5m + 6 = 0$.

or $(m-2)(m-3) = 0$; $\therefore$ $m = 2, 3$.

$\therefore$ $\text{C.F.} = c_1 e^{2x} + c_2 e^{3x}$.

And $$\text{P.I.} = \frac{1}{D^2 - 5D + 6}\sin 3x = \frac{1}{-3^2 - 5D + 6}\sin 3x = \frac{1}{-9 - 5D + 6}\sin 3x$$

$$= \frac{-1}{5D+3}\sin 3x = \frac{-(5D-3)}{(5D+3)(5D-3)}\sin 3x = \frac{-(5D-3)}{25D^2 - 9}\sin 3x$$

$$= \frac{-(5D-3)}{25\cdot(-3^2) - 9}\sin 3x = \frac{1}{234}\{5D(\sin 3x) - 3\sin 3x\}$$

$$= \frac{1}{234}(5\cdot 3\cos 3x - 3\sin 3x) = \frac{1}{78}(5\cos 3x - \sin 3x) .$$

$\therefore$ the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $$y = c_1 e^{2x} + c_2 e^{3x} + (1/78)(5\cos 3x - \sin 3x) .$$

Example 13: *Solve* $\frac{d^2 y}{dx^2} + 9y = \cos 2x + \sin 2x$.

Solution:. The auxiliary equation is

$$m^2 + 9 = 0, \quad \therefore \quad m = \pm 3i .$$

$\therefore$ $\text{C.F.} = c_1 \cos 3x + c_2 \sin 3x$ or $= c_1 \cos(3x + c_2)$

And $$\text{P.I.} = \frac{1}{(D^2+9)}(\cos 2x + \sin 2x) = \frac{1}{D^2+9}\cos 2x + \frac{1}{D^2+9}\sin 2x$$

$$= \frac{\cos 2x}{-2^2+9} + \frac{\sin 2x}{-2^2+9} = \frac{\cos 2x}{5} + \frac{\sin 2x}{5} = \frac{1}{5}(\cos 2x + \sin 2x) .$$

Hence the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $$y = c_1 \cos(3x + c_2) + \frac{1}{5}(\cos 2x + \sin 2x) .$$

Comprehensive Exercise 3

Solve the following differential equations :

1. **(i)** $(d^2 y/dx^2) + 9y = \cos 4x$ **(ii)** $(D^2 + 4)\, y = \sin 3x$ **(Kumaun 2013)**

2. **(i)** $(D^2 - 3D + 2)y = \sin 3x$ **(ii)** $(D^2 - 2D + 5)y = \sin 3x$

(Kumaun 2007; Garhwal 15)

3. **(i)** $(D^2 - 3D + 2)\ y = \cos 3x$ **(ii)** $(D^3 + D^2 - D - 1)\ y = \cos 2x$

(Kanpur 2006; Avadh 08)

4. $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} + y = \cos 3x$ **5.** $\dfrac{d^2 y}{dx^2} - 8\dfrac{dy}{dx} + 9y = 40 \sin 5x$

6. $\dfrac{d^2 y}{dx^2} - 4\dfrac{dy}{dx} + y = a \sin 2x$

(Meerut 2006B)

7. $\dfrac{d^2 y}{dx^2} + 2\dfrac{dy}{dx} + 10y + 37 \sin 2x = 0$

(Lucknow 2005; Gorakhpur 07)

8. $(D^3 + 1)y = \cos 2x$ **9.** $(D^3 - 2D^2 + 3)\ y = \cos x$

Answers 3

1. **(i)** $y = c_1 \cos 3x + c_2 \sin 3x - \dfrac{1}{7} \cos 4x$

(ii) $y = c_1 \cos 2x + c_2 \sin 2x - \dfrac{1}{5} \sin 3x$

2. **(i)** $y = c_1 e^x + c_2 e^{2x} + \dfrac{1}{130}(9 \cos 3x - 7 \sin 3x)$

(ii) $y = e^{2x}(c_1 \cos 2x + c_2 \sin 2x) + (1/26)(3 \cos 3x - 2 \sin 3x)$

3. **(i)** $y = c_1 e^x + c_2 e^{2x} - \dfrac{1}{130}(7 \cos 3x + 9 \sin 3x)$

(ii) $y = (c_1 + c_2 x)\, e^{-x} + c_3 e^x - (1/25)(2 \sin 2x + \cos 2x)$

4. $y = (c_1 + c_2 x)\, e^x - \dfrac{1}{50}(3 \sin 3x + 4 \cos 3x)$

5. $y = c_1 e^{4x} \cosh (x\sqrt{7} + c_2) + \dfrac{5}{29}(5 \cos 5x - 2 \sin 5x)$

6. $y = c_1 e^{2x} \cosh (\sqrt{3}x + c_2) + (1/73)\, a\, (8 \cos 2x - 3 \sin 2x)$

7. $y = e^{-x}(c_1 \cos 3x + c_2 \sin 3x) + \dfrac{37}{26}(2 \cos 2x - 3 \sin 2x)$

8. $y = c_1 e^{-x} + e^{x/2}\left\{c_2 \cos\left(\dfrac{x\sqrt{3}}{2}\right) + c_3 \sin\left(\dfrac{x\sqrt{3}}{2}\right)\right\} + \dfrac{1}{65}(\cos 2x - 8 \sin 2x)$

9. $y = c_1 e^{-x} + e^{3x/2}\left\{c_2 \cos\left(\dfrac{x\sqrt{3}}{2}\right) + c_3 \sin\left(\dfrac{x\sqrt{3}}{2}\right)\right\} + \dfrac{1}{26}(5 \cos x - \sin x)$

Case III: To find P.I. when Q is of the form x^m, where m is a positive integer.

Consider first $\{1/(D - a)\}\ x^m$. We have

$$\frac{1}{(D-a)}x^m=-\frac{1}{(a-D)}x^m=-\frac{1}{a\{1-(D/a)\}}x^m$$
$$=-\frac{1}{a}\left(1-\frac{D}{a}\right)^{-1}x^m$$
$$=-\frac{1}{a}\left(1+\frac{D}{a}+\frac{D^2}{a^2}+\ldots\right)x^m\text{ , expanding by the binomial theorem}$$
$$=-\frac{1}{a}\left\{x^m+\frac{1}{a}mx^{m-1}+\frac{1}{a^2}m(m-1)x^{m-2}+\ldots\right\}.$$

Here we observe that in the expansion by the binomial theorem, the terms of the expansion beyond the mth power of D need not be written since $D^{m+1}x^m=0$, $D^{m+2}x^m=0$, etc.

Working Rule: In order to evaluate $\{1/F(D)\}\,x^m$, bring out common the lowest degree term in D from $F(D)$ so that remaining factor in the denominator is of the form $[1+f(D)]$ or $[1-f(D)]$ which is taken in the numerator with a negative index. Next we expand $[1\pm f(D)]^{-1}$ in powers of D by the binomial theorem and operate upon x^m with the expansion obtained.

This expansion should be done upto the term D^m, since $D^{m+1}x^m=0$ and all higher differential coefficients of x^m are zero. The whole process will be clear from the following examples.

The following binomial expansions should be remembered well.

(i) $(1-x)^{-1}=1+x+x^2+x^3+x^4+\ldots$

(ii) $(1+x)^{-1}=1-x+x^2-x^3+x^4-\ldots$

(iii) $(1-x)^{-2}=1+2x+3x^2+4x^3+\ldots$

(iv) $(1+x)^{-2}=1-2x+3x^2-4x^3+\ldots$

Illustrative Examples

Examples on Case III:

Example 14: *Solve* $(d^2y/dx^2)-4y=x^2$. **(Lucknow 2010)**

Solution: The auxiliary equation is $m^2-4=0$ or $m=\pm 2$.

$\therefore$ C.F. $=c_1e^{2x}+c_2e^{-2x}$.

And
$$\text{P.I.}=\frac{1}{D^2-4}x^2=\frac{1}{-4\,[1-\frac{1}{4}D^2]}x^2=-\frac{1}{4}\left[1-\frac{1}{4}D^2\right]^{-1}x^2$$
$$=-\frac{1}{4}\left[1+\frac{1}{4}D^2+\ldots\right]x^2,$$

expanding by binomial theorem upto the terms containing D^2

$$= -\frac{1}{4}\left[x^2 + \frac{1}{4} D^2 (x^2)\right], \text{ because all the remaining terms vanish}$$

$$= -\frac{1}{4}\left[x^2 + \frac{1}{4}\cdot 2\right] = -\frac{1}{4}\left[x^2 + \frac{1}{2}\right].$$

Hence the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}(x^2 + \frac{1}{2}).$$

Example 15: *Solve* $\dfrac{d^3 y}{dx^3} - \dfrac{d^2 y}{dx^2} - 6\dfrac{dy}{dx} = 1 + x^2$.

(Gorakhpur 2006; Avadh 06)

Solution: The auxiliary equation is

$$m^3 - m^2 - 6m = 0 \quad \text{or} \quad m(m+2)(m-3) = 0.$$

$\therefore$ $m = 0, -2, 3.$

$\therefore$ $\text{C.F.} = c_1 e^{0x} + c_2 e^{-2x} + c_3 e^{3x} = c_1 + c_2 e^{-2x} + c_3 e^{3x}.$

And $$\text{P.I.} = \frac{1}{D^4 - D^2 - 6D}(1 + x^2) = \frac{1}{-6D[1 + \frac{1}{6}D - \frac{1}{6}D^2]}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \frac{1}{6}(-D + D^2)\right]^{-1}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 + \frac{1}{6}(-D + D^2) + \frac{1}{36}(-D + D^2)^2 + \ldots\right](1 + x^2),$$

the terms in the expansion being needed only upto D^2

$$= -\frac{1}{6D}\left[1 - \frac{1}{6}D + \frac{1}{6}D^2 + \frac{1}{36}D^2 + \ldots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \frac{1}{6}D + \frac{7}{36}D^2 + \ldots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[(1 + x^2) - \frac{1}{6}D(1 + x^2) + \frac{7}{36}D^2(1 + x^2)\right],$$

because all other terms vanish

$$= -\frac{1}{6D}\left(1 + x^2 - \frac{1}{3}x + \frac{7}{18}\right) = -\frac{1}{6D}\left(\frac{25}{18} - \frac{1}{3}x + x^2\right)$$

$$= -\frac{1}{6}\int\left(\frac{25}{18} - \frac{1}{3}x + x^2\right)dx, \qquad \left[\because\ 1/D \equiv \int dx\right]$$

$$= -\frac{1}{6}\left[\frac{25}{18}x - \frac{1}{6}x^2 + \frac{1}{3}x^3\right].$$

Hence the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $$y = c_1 + c_2 e^{-2x} + c_3 e^{3x} - \frac{1}{6}\left[\frac{25}{18}x - \frac{1}{6}x^2 + \frac{1}{3}x^3\right].$$

Example 16: *Solve* $(D^3 + 8)\, y = x^4 + 2x + 1$.

Solution: The auxiliary equation is

$$m^3+8=0 \qquad \text{or} \qquad (m+2)(m^2-2m+4)=0.$$

$$\therefore \qquad m=-2, 1\pm i\sqrt{3}$$

Hence C.F. $= c_1 e^{-2x} + e^x \{c_2 \cos(x\sqrt{3}) + c_3 \sin(x\sqrt{3})\}$.

And $$\text{P.I.} = \frac{1}{D^3+8}(x^4+2x+1) = \frac{1}{8\left[1+\frac{1}{8}D^3\right]}(x^4+2x+1)$$

$$= \frac{1}{8}[1+\tfrac{1}{8}D^3]^{-1}(x^4+2x+1)$$

$$= \frac{1}{8}[1-\tfrac{1}{8}D^3+\ldots]\,(x^4+2x+1),$$

the other terms in the expansion being of no need

$$= \frac{1}{8}[(x^4+2x+1)] - \frac{1}{8}D^3\,(x^4+2x+1)$$

$$= \frac{1}{8}[x^4+2x+1-3x] = \frac{1}{8}(x^4-x+1).$$

Hence the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $$y = c_1 e^{-2x} + e^x\{c_2\cos(x\sqrt{3}) + c_3\sin(x\sqrt{3})\} + \frac{1}{8}(x^4-x+1).$$

Example 17: *Solve* $(D^2+D-2)\,y = x+\sin x$.

Solution: The auxiliary equation is $m^2+m-2=0$

or $$(m-1)(m+2)=0.$$

$$\therefore \qquad m=1,-2. \qquad \therefore \qquad \text{C.F.} = c_1e^x + c_2e^{-2x}.$$

And $$\text{P.I.} = \frac{1}{D^2+D-2}(x+\sin x) = \frac{1}{(D^2+D-2)}x + \frac{1}{D^2+D-2}\sin x$$

$$= \frac{1}{-2\left[1-\frac{1}{2}D-\frac{1}{2}D^2\right]}x + \frac{1}{-1^2+D-2}\sin x$$

$$= -\frac{1}{2}\left[1-\left(\frac{1}{2}D+\frac{1}{2}D^2\right)\right]^{-1}x + \frac{(D+3)}{(D-3)(D+3)}\sin x$$

$$= -\frac{1}{2}\left[1+\frac{1}{2}D+\ldots\right]x + \frac{(D+3)}{D^2-9}\sin x = -\frac{1}{2}\left(x+\frac{1}{2}\right) + \frac{D+3}{-1-9}\sin x$$

$$= -\frac{1}{2}(x+\frac{1}{2}) - \frac{1}{10}\{D(\sin x)+3\sin x\}.$$

$$= -\frac{1}{2}x - \frac{1}{4} - \frac{1}{10}\{\cos x + 3\sin x\}.$$

Hence the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $$y = c_1e^x + c_2e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{10}\cos x - \frac{3}{10}\sin x.$$

Example 18: *Solve* $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = x^2 + e^x + \cos 2x$.

(Meerut 2001)

Solution: The auxiliary equation is $m^2 - 4m + 4 = 0$.

$\therefore$ $\quad m = 2, 2$.

Hence $\quad \text{C.F.} = (c_1 + c_2 x)\, e^{2x}$,

and $\quad \text{P.I.} = \dfrac{1}{(D^2 - 4D + 4)} (x^2 + e^x + \cos 2x) = P_1 + P_2 + P_3$, (say),

where
$$P_1 = \frac{1}{D^2 - 4D + 4} x^2 = \frac{1}{(D-2)^2} x^2 = \frac{1}{4}\left[1 - \frac{1}{2} D\right]^{-2} x^2$$
$$= \frac{1}{4}\left[1 + D + \frac{3}{4} D^2 + \dots\right] x^2 = \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right],$$
$$P_2 = \frac{1}{(D^2 - 4D + 4)} e^x = \frac{e^x}{1^2 - 4 \cdot 1 + 4} = \frac{e^x}{1} = e^x,$$
and
$$P_3 = \frac{1}{(D^2 - 4D + 4)]} \cos 2x = \frac{1}{-2^2 - 4D + 4} \cos 2x = -\frac{1}{4D} \cos 2x$$
$$= -\frac{1}{4} \int \cos 2x \, dx = -\frac{1}{4} \cdot \frac{1}{2} \sin 2x = -\frac{1}{8} \sin 2x.$$

(Rohilkhand 2008)

Hence the complete solution is $y = \text{C.F.} + \text{P.I.}$

or $\quad y = \text{C.F.} + P_1 + P_2 + P_3$

or $\quad y = (c_1 + c_2 x)\, e^{2x} + \frac{1}{4} (x^2 + 2x + \frac{3}{2}) + e^x - \frac{1}{8} \sin 2x.$

Comprehensive Exercise 4

1. $(D^2 + D - 6)\, y = x$. **(Bundelkhand 2007)**

2. $(d^3 y/dx^3) + 3(d^2 y/dx^2) + 2(dy/dx) = x^2$. **(Bundelkhand 2001)**

3. $\dfrac{d^3 y}{dx^3} - 4 \dfrac{d^2 y}{dx^2} + 5 \dfrac{dy}{dx} - 2 = 0$.

4. $(D^2 + 2D + 1)\, y = 2x + x^2$. **(Kumaun 2007)**

5. $(D^2 + 3D + 2)\, y = x^2$. **(Bundelkhand 2001)**

6. $(D^4 + D^2 + 16)\, y = 16x^2 + 256$. **(Lucknow 2005)**

7. $(D^2 - 5D + 6)\, y = x + \sin 3x$. **(Kanpur 2007)**

8. $(D^2 - 4D + 3)\, y = e^{-x} + 5$.

9. $(d^2 y/dx^2) - 4y = \sin^2 x$. **(Gorakhpur 2007)**

10. $(D^2 - 2D + 3)\, y = \cos x + x^2$. **(Meerut 2009B; Bundelkhand 11)**

11. $\dfrac{d^3 y}{dx^3} + 2\dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} = e^{2x} + x^2 + x$. **(Meerut 2009; Gorakhpur 10)**

12. $(D^4 - 2D^3 + D^2)\, y = x^3$. **(Garhwal 2007)**

Answers 4

1. $y = c_1 e^{2x} + c_2 e^{-3x} - \dfrac{1}{36}(6x + 1)$

2. $y = c_1 + c_2 e^{-x} + c_3 e^{-2x} + \dfrac{1}{12} x\,(2x^2 - 9x + 21)$

3. $y = c_1 + c_2 e^{2x} \cos(x + c_3) + \dfrac{2}{5} x$

4. $y = (c_1 + c_2 x)\, e^{-x} + (x^2 - 2x + 2)$

5. $y = c_1 e^{-x} + c_2 e^{-2x} + \dfrac{1}{2} x^2 - \dfrac{3}{2} x + \dfrac{7}{4}\cdot$

6. $y = c_1 e^{-(1/2)\sqrt{7}.x} \sin\left(\dfrac{3}{2} x + c_2\right) + c_3 e^{(1/2)\sqrt{7}x} \sin\left(\dfrac{3}{2} x + c_4\right) + x^2 + \dfrac{127}{8}$

7. $y = c_1 e^{2x} + c_2 e^{3x} + \dfrac{1}{6}\left(x + \dfrac{5}{6}\right) - \dfrac{1}{234}(3 \sin 3x - 15 \cos 3x)$

8. $y = c_1 e^{x} + c_2 e^{3x} + \dfrac{1}{8} e^{-x} + \dfrac{5}{3}$

9. $y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{1}{8} + \dfrac{1}{16} \cos 2x$

10. $y = e^{x}\, [c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x] + \dfrac{1}{3} x^2 + \dfrac{4}{9} x + \dfrac{2}{27} + \dfrac{1}{4}(\cos x - \sin x)$

11. $y = c_1 + (c_2 + c_3 x)\, e^{-x} + \dfrac{1}{18} e^{2x} + \dfrac{1}{3} x^3 - \dfrac{3}{2} x^2 + 4x$

12. $y = c_1 + c_2 x + (c_3 + c_4 x)\, e^{x} + \dfrac{1}{20} x^5 + \dfrac{1}{2} x^4 + 3x^3 + 36x^2$

5 P.I. when $Q = e^{ax} V$, where V is any Function of x

By successive differentiation we notice that

$$D\,(e^{ax} V) = e^{ax}\, D\,(V) + V a e^{ax} = e^{ax}\,(D + a)\, V,$$

$$D^2\,(e^{ax} V) = D\{e^{ax}\,(D + a)\, V\}$$

$$= e^{ax}\, D\,(D + a)\, V + a e^{ax}\,(D + a)\, V = e^{ax}\,(D + a)^2\, V.$$

Similarly, $D^3\,(e^{ax} V) = e^{ax}\,(D + a)^3\, V, \ldots,$

$$D^n (e^{ax} V) = e^{ax} (D+a)^n V .$$

$$\therefore \qquad F(D)(e^{ax} V) = e^{ax} F(D+a) V . \qquad \ldots(1)$$

The result (1) is true for any function V of x. Taking $\dfrac{1}{F(D+a)} V$ in place of V in (1), we have

$$F(D)\left\{e^{ax} \frac{1}{F(D+a)} V\right\} = e^{ax} F(D+a) \left\{\frac{1}{F(D+a)} V\right\}$$

i.e.,
$$F(D)\left\{e^{ax} \frac{1}{F(D+a)} V\right\} = e^{ax} V . \qquad \ldots(2)$$

Operating by $\dfrac{1}{F(D)}$ on both sides of (2), we get

$$e^{ax} \frac{1}{F(D+a)} V = \frac{1}{F(D)} (e^{ax} V).$$

Thus,
$$\frac{1}{F(D)} (e^{ax} V) = e^{ax} \frac{1}{F(D+a)} V.$$

Working Rule: Replace D by $(D+a)$ and take out e^{ax} before the operator $1/F(D)$. Then determine $\{1/F(D+a)\} V$ by the methods discussed in 4.

This method also enables us to find $\{1/F(D)\} e^{ax}$ when $F(a)$ is zero. We shall discuss it later on in article 6.

Illustrative Examples

Example 19: *Solve* $(D^2 - 2D + 1)\, y = x^2 e^{3x}$.

(Agra 2005; Kumaun 06. 08; Avadh 09; Garhwal 13)

Solution: The auxiliary equation is

$$m^2 - 2m + 1 = 0 \qquad \text{or} \qquad (m-1)^2 = 0.$$

$$\therefore \qquad m = 1, 1 .$$

$$\therefore \qquad \text{C.F.} = (c_1 + c_2 x)\, e^x .$$

And
$$\text{P. I.} = \frac{1}{D^2 - 2D + 1} x^2 e^{3x} = \frac{1}{(D-1)^2} e^{3x} x^2$$

$$= e^{3x} \frac{1}{\{(D+3) - 1\}^2} x^2,$$

putting $D + 3$ for D and bringing e^{3x} before the operator

$$= e^{3x} \frac{1}{(D+2)^2} x^2 = e^{3x} \frac{1}{4\left(1 + \frac{1}{2} D\right)^2} x^2 = \frac{1}{4} e^{3x} \left(1 + \frac{1}{2} D\right)^{-2} x^2$$

$$=\frac{1}{4}e^{3x}\left[1-2\cdot\frac{1}{2}D+3\cdot\frac{1}{4}D^2+\ldots\right]x^2,$$

expanding by binomial theorem

$$=\frac{1}{4}e^{3x}\left[1-D+\frac{3}{4}D^2+\ldots\right]x^2$$

$$=\frac{1}{4}e^{3x}\left\{x^2-2x+\frac{3}{4}\cdot 2\right\}=\frac{1}{4}e^{3x}\left\{x^2-2x+\frac{3}{2}\right\}.$$

Hence the required solution is $y=(\text{C.F.})+(\text{P.I.})$

or $$y=(c_1+c_2x)\,e^x+\frac{1}{4}e^{3x}\left\{x^2-2x+\frac{3}{2}\right\}.$$

Example 20: *Solve* $(D^2-2D+5)\,y=e^{2x}\sin x.$

Solution: The auxiliary equation is $m^2-2m+5=0$.

$\therefore$ $m=\{2\pm\sqrt{(4-20)}\}/2=1\pm 2i.$

Hence C.F. $=e^x\,(c_1\cos 2x+c_2\sin 2x)$.

And $$\text{P.I.}=\frac{1}{D^2-2D+5}e^{2x}\sin x$$

$$=e^{2x}\frac{1}{(D+2)^2-2\,(D+2)+5}\sin x=e^{2x}\frac{1}{D^2+2D+5}\sin x$$

$$=e^{2x}\frac{1}{(-1^2+2D+5)}\sin x,\ \text{putting }-1^2\text{ for }D^2$$

$$=\frac{1}{2}e^{2x}\frac{1}{D+2}\sin x=\frac{1}{2}e^{2x}\frac{D-2}{(D-2)\,(D+2)}\sin x$$

$$=\frac{1}{2}e^{2x}\frac{D-2}{(D^2-4)}\sin x=\frac{1}{2}e^{2x}\frac{D-2}{-1-4}\sin x$$

$$=-\frac{1}{10}e^{2x}\,(D-2)\sin x=-\frac{1}{10}e^{2x}\,\{D\,(\sin x)-2\sin x\}$$

$$=-\frac{1}{10}e^{2x}\,(\cos x-2\sin x).$$

Hence the complete solution is $y=(\text{C.F.})+(\text{P.I.})$

or $$y=e^x\,(c_1\cos 2x+c_2\sin 2x)-\frac{1}{10}e^{2x}\,(\cos x-2\sin x).$$

Comprehensive Exercise 5

Solve the following differential equations :

1. $(D+1)^3\ y=x^2\,e^{-x}.$
2. $(D^2-2D+1)\,y=x^2\,e^x.$ **(Avadh 2007; Kumaun 09)**

3. $(D^2+2D+1)\,y=e^{-x}/(x+2)$.
4. $(D^2-3D+2)\,y=xe^x$. **(Bundelkhand 2008)**
5. $(D^2-2D+1)\,y=x^2\,e^{2x}$. **(Purvanchal 2007)**
6. $(D^2-4D+4)\,y=e^{2x}\sin 3x$. **(Purvanchal 2006)**
7. $(D^4-1)\,y=e^x\cos x$. **(Lucknow 2010)**
8. $(D^2-2D+4)\,y=e^x\cos x$. **(Garhwal 2001)**
9. $(D^2+4D-12)\,y=(x-1)\,e^{2x}$.
10. $(D^2-5D+6)\,y=e^{2x}\sin 2x$.
11. $(D^2-4D+3)y=e^{2x}\sin 3x$. **(Garhwal 2008)**
12. $(D^2+2D+2)\,y=xe^x$.
13. $(D^2-1)\,y=e^x\cos x$.

Answers 5

1. $y=(c_1+c_2x+c_3x^2)\,e^{-x}+\dfrac{1}{60}\,x^5\,e^{-x}$
2. $y=(c_1+c_2x)\,e^x+\dfrac{1}{12}\,x^4e^x$
3. $y=(c_1+c_2x)\,e^{-x}+e^{-x}\{x\log(x+2)-x+2\log(x+2)\}$
4. $y=c_1e^x+c_2e^{2x}-e^x\left(\dfrac{1}{2}x^2+x\right)$
5. $y=(c_1+c_2x)\,e^x+e^{2x}\,(x^2-4x+6)$
6. $y=(c_1+c_2x)\,e^{2x}-\dfrac{1}{9}\,e^{2x}\sin 3x$
7. $y=c_1\,e^x+c_2e^{-x}+c_3\cos x+c_4\sin x-\dfrac{1}{5}\,e^x\cos x$
8. $y=e^x\,(c_1\cos\sqrt{}3x+c_2\sin\sqrt{}3x)+\dfrac{1}{2}\,e^x\cos x$
9. $y=e^{2x}\,(c_1+\dfrac{1}{16}\,x^2-\dfrac{9}{64}\,x)+c_2\,e^{-6x}$
10. $y=c_1e^{2x}+c_2e^{3x}+\dfrac{1}{10}\,e^{2x}\,(\cos 2x-2\sin 2x)$
11. $y=c_1e^x+c_2e^{3x}-\dfrac{1}{10}\,e^{2x}\sin 3x$
12. $y=e^{-x}(c_1\cos x+c_2\sin x)+\dfrac{1}{5}\,e^x\left(x-\dfrac{4}{5}\right)$
13. $y=c_1e^x+c_2e^{-x}-\dfrac{1}{5}\,e^x\,(\cos x-2\sin x)$

6 P.I. when $Q = e^{ax}$ and $F(a) = 0$

In this case we use the method explained in article 5 to find the P.I.

We have **P.I.** $= \dfrac{\mathbf{1}}{\boldsymbol{F(D)}}\, \boldsymbol{e^{ax}} = \dfrac{\mathbf{1}}{\boldsymbol{F(D)}}\, \boldsymbol{e^{ax}}.\,1$, if $F(a) = 0$.

Now this is of the form $e^{ax}\,V$. Hence by the method of article 5, *i.e.*, by putting $(D + a)$ for D in the operator and by making e^{ax} free from the operator, we have

$$\text{P.I.} = e^{ax}\,\frac{1}{F(D+a)}\,1.$$

Now this can be evaluated by using the method for finding P.I. in the case of x^m.

Illustrative Examples

Example 21: *Solve* $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^x$. **(Garhwal 2003, 05)**

Solution: The auxiliary equation is $m^2 - 3m + 2 = 0$

or $(m-1)(m-2) = 0$. $\therefore$ $m = 1, 2$.

Hence $\text{C.F.} = c_1e^x + c_2e^{2x}$.

And $$\text{P.I.} = \frac{1}{D^2 - 3D + 2}e^x = \frac{1}{(D-2)(D-1)}e^x = \frac{1}{(1-2)(D-1)}e^x$$

[putting 1 for D in the factor $D - 2$ because it does not vanish by doing so]

$$= -\frac{1}{D-1}e^x \cdot 1$$

[**Note** that $D - 1$ becomes zero by putting 1 for D; so here we shall apply the method for $e^{ax}V$ by taking 1 for V.]

$$= -e^x\frac{1}{(D+1)-1}1 = -e^x\frac{1}{D}1 = -e^x x.$$

Hence the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $y = c_1e^x + c_2e^{2x} - x\,e^x$.

Note: While finding P.I. in the case of e^{ax} if $F(D)$ becomes zero by putting a for D, we factorise $F(D)$. Then we put $D = a$ in the factors which do not vanish by doing so. The remaining operator is then dealt with by using the method for $e^{ax}V$ on taking 1 for V.

Example 22: *Solve* $(D^3 - 7D + 6)\,y = e^{2x}$.

Solution: The auxiliary equation is

$$m^3 - 7m + 6 = 0 \quad \text{or} \quad (m-1)(m-2)(m+3) = 0.$$

$\therefore$ $m = 1, 2, -3$.

$\therefore$ C. F. $= c_1 e^{x} + c_2 e^{2x} + c_3 e^{-3x}$.

And P. I. $= \dfrac{1}{(D^3 - 7D + 6)} e^{2x} = \dfrac{1}{(D-1)(D-2)(D+3)} e^{2x}$

$= \dfrac{1}{(2-1)(D-2)(2+3)} e^{2x}$,

putting 2 for D in all the factors except $D - 2$

$= \dfrac{1}{5} \dfrac{1}{D-2} e^{2x} \cdot 1 = \dfrac{1}{5} e^{2x} \dfrac{1}{(D+2)-2} 1 = \dfrac{1}{5} e^{2x} \dfrac{1}{D} 1 = \dfrac{1}{5} x e^{2x}$.

Hence the complete solution is

$y =$ (C. F.) + (P. I.) or $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{-3x} + \dfrac{1}{5} x e^{2x}$.

Example 23: *Solve* $(D^2 + 4D + 4)\, y = e^{2x} - e^{-2x}$

or $(D^2 + 4D + 4)\, y = 2 \sinh 2x$. **(Garhwal 2010)**

Solution: The auxiliary equation is $(m+2)^2 = 0$.

$\therefore$ $m = -2, -2$.

Hence C. F. $= (c_1 + c_2 x)\, e^{-2x}$.

And P. I. $= \dfrac{1}{(D+2)^2} (e^{2x} - e^{-2x}) = \dfrac{1}{(D+2)^2} e^{2x} - \dfrac{1}{(D+2)^2} e^{-2x}$.

Now $\dfrac{1}{(D+2)^2} e^{2x} = \dfrac{e^{2x}}{(2+2)^2} = \dfrac{1}{16} e^{2x}$. [$\because$ Here, $F(a) \neq 0$]

Also $\dfrac{1}{(D+2)^2} e^{-2x} = \dfrac{1}{(D+2)^2} e^{-2x} \cdot 1$, [$\because$ Here $F(a) = 0$]

$= e^{-2x} \dfrac{1}{\{(D-2)+2\}^2} 1 = e^{-2x} \dfrac{1}{D^2} 1 = e^{-2x} \dfrac{1}{D} x = \dfrac{1}{2} x^2 e^{-2x}$.

Hence the complete solution is $y =$ (C. F.) + (P. I.)

or $y = (c_1 + c_2 x)\, e^{-2x} + \dfrac{1}{16} e^{2x} - \dfrac{1}{2} x^2 e^{-2x}$.

Example 24: *Solve* $D^2 y - 3Dy + 2y = \cosh x$. **(Rohilkhand 2010)**

Solution: The auxiliary equation is

$m^2 - 3m + 2 = 0$ or $(m-1)(m-2) = 0$ $\therefore$ $m = 1, 2$.

$\therefore$ C. F. $= c_1 e^{x} + c_2 e^{2x}$.

And P. I. $= \dfrac{1}{D^2 - 3D + 2} \cosh x = \dfrac{1}{(D^2 - 3D + 2)} \left(\dfrac{e^{x} + e^{-x}}{2} \right)$

$= \dfrac{1}{2} \dfrac{1}{D^2 - 3D + 2} e^{x} + \dfrac{1}{2} \dfrac{1}{D^2 - 3D + 2} e^{-x}$.

Now $$\frac{1}{2}\frac{1}{D^2-3D+2}e^x=\frac{1}{2}\frac{1}{(D-1)(D-2)}e^x$$

$$=\frac{1}{2}\frac{1}{(D-1)(1-2)}e^x=-\frac{1}{2}\frac{1}{D-1}e^x\cdot 1$$

$$=-\frac{1}{2}e^x\frac{1}{D+1-1}1=-\frac{1}{2}e^x\frac{1}{D}1=-\frac{1}{2}xe^x.$$

Also $$\frac{1}{2}\frac{1}{D^2-3D+2}e^{-x}=\frac{1}{2}\frac{1}{(-1)^2-3(-1)+2}e^{-x}, \text{ putting } -1 \text{ for } D$$

$$=\frac{1}{12}e^{-x}.$$

$\therefore$ $$\text{P.I.}=-\frac{1}{2}xe^x+\frac{1}{12}e^{-x}.$$

Hence the complete solution is $y=(\text{C.F.})+(\text{P.I.})$

or $$y=c_1e^x+c_2e^{2x}-\frac{1}{2}xe^x+\frac{1}{12}e^{-x}.$$

Example 25: *Solve* $(D^2-4D+4)\,y=e^{2x}+\sin 2x$.

Solution: The auxiliary equation is

$$m^2-4m+4=0 \quad \text{or} \quad (m-2)^2=0.$$

$\therefore$ $m=2,2$.

Hence $\text{C.F.}=(c_1+c_2x)\,e^{2x}$.

And $$\text{P.I.}=\frac{1}{(D-2)^2}e^{2x}+\frac{1}{D^2-4D+4}\sin 2x.$$

Now $$\frac{1}{(D-2)^2}e^{2x}=\frac{1}{(D-2)^2}e^{2x}\cdot 1=e^{2x}\frac{1}{(D+2-2)^2}1$$

$$=e^{2x}\frac{1}{D^2}(1)=e^{2x}\frac{1}{D}(x)=e^{2x}\cdot\frac{x^2}{2}=\frac{1}{2}x^2e^{2x},$$

and $$\frac{1}{D^2-4D+4}\sin 2x=\frac{1}{-2^2-4D+4}\sin 2x, \text{ putting } -2^2 \text{ for } D^2$$

$$=\frac{1}{-4D}\sin 2x=-\frac{1}{4}\int\sin 2x\,dx=-\frac{1}{4}\cdot\frac{1}{2}(-\cos 2x)=\frac{1}{8}\cos 2x.$$

Hence the complete solution is $y=(\text{C.F.})+(\text{P.I.})$

or $$y=(c_1+c_2x)\,e^{2x}+\frac{1}{2}x^2e^{2x}+\frac{1}{8}\cos 2x.$$

Comprehensive Exercise 6

Solve the following differential equations :

1. $(D^2-2D+1)\,y=e^x$. **(Garhwal 2006)**
2. $(D^2+D-6)\,y=e^{2x}$.

3. $(D^2 - 4D + 3)\ y = 2e^{3x}$.
4. $(D^3 + 3D^2 + 3D + 1)\ y = e^{-x}$.
5. $(D^2 - a^2) = \cosh ax$.
6. $(D^3 - 5D^2 + 7D - 3)\ y = e^{2x} \cosh x$. **(Rohilkhand 2005)**
7. $(D^2 - 4D + 4)\ y = 8\,(x^2 + e^{2x} + \sin 2x)$.

Answers 6

1. $y = (c_1 + c_2 x)\, e^x + \frac{1}{2} x^2 e^x$
2. $y = c_1 e^{2x} + c_2 e^{-3x} + \frac{1}{5} x e^{2x}$
3. $y = c_1 e^x + c_2 e^{3x} + x e^{3x}$
4. $y = (c_1 + c_2 x + c_3 x^2)\, e^{-x} + \frac{1}{6} x^3 e^{-x}$
5. $y = c_1 e^{ax} + c_2 e^{-ax} + (x/2a) \sinh ax$
6. $y = (c_1 + c_2 x)\, e^x + c_3\, e^{3x} + \frac{1}{8} x e^{3x} - \frac{1}{8} x^2 e^x$
7. $y = (c_1 + c_2 x)\, e^{2x} + 2\,(x^2 + 2x + \frac{3}{2}) + 4x^2 e^{2x} + \cos 2x$

7 P.I. when $Q = \sin ax$ or $\cos ax$ and $F(-a^2) = 0$

To find out these types of particular integrals, it is convenient to replace $\sin ax$ or $\cos ax$ by the exponential value and then apply 6. Thus the particular integral of $\cos ax$

i.e., $\frac{1}{F(D^2)} \cos ax$, when $F(-a^2) = 0$, can be written as

$$= \frac{1}{F(D^2)} \text{(Real part of } e^{iax}\text{)}, \qquad [\because e^{iax} = \cos ax + i \sin ax]$$

$$= \text{Real Part of } \frac{1}{F(D^2)} e^{iax} \cdot 1, \text{ which is now of the form of article 5.}$$

Similarly, the particular integral of $\sin ax$

i.e., $\frac{1}{F(D^2)} \sin ax$, when $F(-a^2) = 0$

$$= \frac{1}{F(D^2)} \text{(Imaginary part of } e^{iax}\text{)}, \qquad [\because e^{iax} = \cos ax + i \sin ax]$$

$$= \textbf{Imaginary part of } \frac{1}{F(D^2)} e^{iax} \cdot 1,.$$

which is now of the form $\frac{1}{F(D)} e^{ax} V$ of article 5.

Now the method of article 5 is to be followed.

Illustrative Examples

Example 26: *Solve* $\frac{d^2 y}{dx^2} + a^2 y = \cos ax$.

(Meerut 2003)

Solution: The auxiliary equation is

$$m^2 + a^2 = 0, \text{ i.e., } m = 0 \pm ai.$$

$$\therefore \quad \text{C.F.} = e^{0x}(c_1 \cos ax + c_2 \sin ax)$$
$$= c_1 \cos ax + c_2 \sin ax.$$

And
$$\text{P.I.} = \frac{1}{D^2 + a^2} \cos ax$$
$$= \text{the real part in } \frac{1}{D^2 + a^2} (\cos ax + i \sin ax)$$
$$= \text{the real part in } \frac{1}{D^2 + a^2} e^{iax},$$

by Euler's theorem of trigonometry.

Now
$$\frac{1}{D^2 + a^2} e^{iax} = \frac{1}{(D + ia)(D - ia)} e^{iax}$$
$$= \frac{1}{(ia + ia)(D - ia)} e^{iax}, \text{ putting } ia \text{ for } D \text{ in the factor } D + ia$$
$$= \frac{1}{2ia(D - ia)} e^{iax} \cdot 1$$
$$= \frac{1}{2ia} e^{iax} \frac{1}{(D + ia) - ia} 1 = \frac{1}{2ia} e^{iax} \frac{1}{D} 1$$
$$= \frac{1}{2ia} e^{iax} . x = \frac{x}{2ai} (\cos ax + i \sin ax), \qquad [\because e^{iax} = \cos ax + i \sin ax]$$
$$= -\frac{ix}{2a} (\cos ax + i \sin ax), \qquad \left[\because \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i\right]$$
$$= -i \frac{x}{2a} \cos ax + \frac{x}{2a} \sin ax, \qquad [\because i^2 = -1]$$

$$\therefore \quad \frac{1}{D^2 + a^2} \cos ax = \text{the real part in} \left(\frac{x}{2a} \sin ax - i \frac{x}{2a} \cos ax\right)$$
$$= \frac{x}{2a} \sin ax.$$

(Agra 2006)

Hence the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $\quad y = c_1 \cos ax + c_2 \sin ax + (x/2a) \sin ax.$

Remember: While solving the differential equation $(D^2 + a^2) y = \cos ax$, the P.I.

$$\mathbf{\frac{1}{D^2 + a^2} \cos ax = \frac{x}{2a} \sin ax = \frac{x}{2} \int \cos ax \, dx}.$$

Example 27: *Solve* $(D^2 + a^2)\, y = \sin ax$. **(Bundelkhand 2008; Rohilkhand 10)**

Solution: Here as in Example 26, C.F. $= c_1 \cos ax + c_2 \sin ax$.

Also $\quad \text{P.I.} = \dfrac{1}{D^2 + a^2} \sin ax$

$= \text{the coefficient of } i \text{ in } \dfrac{1}{D^2 + a^2} (\cos ax + i \sin ax)$

$= \text{the coefficient of } i \text{ in } \dfrac{1}{D^2 + a^2} e^{iax}$.

Now $\quad \dfrac{1}{D^2 + a^2} e^{iax} = -i \dfrac{x}{2a} \cos ax + \dfrac{x}{2a} \sin ax$. [Proceed as in Example 26]

$\therefore \quad \dfrac{1}{D^2 + a^2} \sin ax$

$= \text{the coefficient of } i \text{ in } \left(-i \dfrac{x}{2a} \cos ax + \dfrac{x}{2a} \sin ax \right) = -\dfrac{x}{2a} \cos ax$.

Hence the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $\quad y = c_1 \cos ax + c_2 \sin ax - (x/2a) \cos ax$.

Remember : $\boldsymbol{\dfrac{1}{D^2 + a^2} \sin ax = -\dfrac{x}{2a} \cos ax = \dfrac{x}{2} \int \sin ax \, dx}$ **.**

Example 28: *Solve* $(D^2 + 4)\, y = \sin^2 x$.

(Rohilkhand 2007; Gorakhpur 10; Avadh 11)

Solution: The auxiliary equation is $m^2 + 2 = 0$.

$\therefore \quad m = 0 \pm 2i$.

Hence $\quad \text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$. $\quad [\because e^{0x} = 1]$

And $\quad \text{P.I.} = \dfrac{1}{D^2 + 4} \sin^2 x = \dfrac{1}{D^2 + 4} \dfrac{1}{2} (2 \sin^2 x) = \dfrac{1}{2} \dfrac{1}{(D^2 + 4)} (1 - \cos 2x)$

$= \dfrac{1}{2} \dfrac{1}{D^2 + 4} 1 - \dfrac{1}{2} \dfrac{1}{D^2 + 4} \cos 2x$.

Now $\quad \dfrac{1}{2} \dfrac{1}{D^2 + 4} 1 = \dfrac{1}{8} \left(1 + \dfrac{1}{4} D^2 \right)^{-1} 1 = \dfrac{1}{8} \left(1 - \dfrac{1}{4} D^2 + \ldots \right) 1 = \dfrac{1}{8}$.

Again $\quad \dfrac{1}{2} \dfrac{1}{D^2 + 4} \cos 2x = \dfrac{1}{2} \cdot \dfrac{1}{4} x \sin 2x$. [Refer Example 26]

Hence the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $\quad y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{1}{8} - \dfrac{1}{8} x \sin 2x$.

Example 29: *Solve* $(D^2 + 1)\, y = \sin x \sin 2x$.

Solution: The auxiliary equation is $m^2 + 1 = 0$ or $m = \pm i$.

$\therefore \quad \text{C.F.} = c_1 \cos x + c_2 \sin x$.

And $$\text{P.I.} = \frac{1}{D^2+1}(\sin x \sin 2x) = \frac{1}{D^2+1}\frac{1}{2}(2\sin x \sin 2x)$$

$$= \frac{1}{2}\frac{1}{D^2+1}(\cos x - \cos 3x) = \frac{1}{2}\frac{1}{D^2+1}\cos x - \frac{1}{2}\frac{1}{D^2+1}\cos 3x.$$

Now $$\frac{1}{D^2+1}\cos x = \text{the real part in } \frac{1}{D^2+1}e^{ix}$$

$$= \text{the real part in } \frac{1}{(D+i)(D-i)}e^{ix}$$

$$= \text{the real part in } \frac{1}{(i+i)(D-i)}e^{ix}$$

$$= \text{the real part in } \frac{1}{2i(D-i)}e^{ix}\cdot 1 = \text{the real part in } \frac{1}{2i}e^{ix}\frac{1}{D+i-i}1$$

$$= \text{the real part in } \frac{1}{2i}e^{ix}\frac{1}{D}1 = \text{the real part in } \frac{1}{2i}e^{ix}x$$

$$= \text{the real part in } \frac{x}{2i}(\cos x + i\sin x)$$

$$= \text{the real part in } -\frac{1}{2}xi(\cos x + i\sin x)$$

$$= \text{the real part in } -i\frac{1}{2}x\cos x + \frac{1}{2}x\sin x = \frac{1}{2}x\sin x.$$

Again $$\frac{1}{D^2+1}\cos 3x = \frac{1}{-3^2+1}\cos 3x = -\frac{1}{8}\cos 3x.$$

$\therefore$ $$\text{the P.I.} = \frac{1}{2}\cdot\frac{1}{2}x\sin x - \frac{1}{2}\cdot\left(-\frac{1}{8}\cos 3x\right) = \frac{1}{4}x\sin x + \frac{1}{16}\cos 3x.$$

Hence the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $$y = c_1\cos x + c_2\sin x + \frac{1}{4}x\sin x + \frac{1}{16}\cos 3x.$$

Comprehensive Exercise 7

Solve the following differential equations :

1. **(i)** $(D^2+4)\,y = \sin 2x.$ **(Agra 2005; Gorakhpur 09; Kumaun 11)**

   (ii) $(D^2+4)\,y = \cos 2x.$ **(Kumaun 2012)**

2. $(D^3 + a^2D)\,y = \sin ax.$ **(Avadh 2005, Gorakhpur 05)**
3. $(D^2+9)(D^2+1)\,y = \cos 3x.$
4. $(D^4 - m^4)\,y = \sin mx.$
5. $(D^2+9)\,y = 2\sin 3x + \cos 3x.$
6. $(D^4 + 2a^2D^2 + a^4)\,y = \cos ax.$

7. Find the solution of the equation $\frac{d^2y}{dx^2} + 4y = 8\cos 2x$, given that $y = 0$ and $\frac{dy}{dx} = 2$ when $x = 0$.

Answers 7

1. **(i)** $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} x \cos 2x$

   (ii) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4} \sin 2x$

2. $y = c_1 + c_2 \cos ax + c_3 \sin ax - (1/2a^2)x \sin ax$
3. $y = c_1 \cos 3x + c_2 \sin 3x + c_3 \cos x + c_4 \sin x - \frac{1}{48} x \sin 3x$
4. $y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx + \frac{x}{4m^3} \cos mx$
5. $y = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{3} x \cos 3x + \frac{1}{6} x \sin 3x.$
6. $y = (c_1 + c_2 x)\cos ax + (c_3 + c_4 x)\sin ax - \frac{x^2}{8a^2} \cos ax$
7. $y = \sin 2x + 2x \sin 2x$

8 P.I. when $Q = xV$, where V is any Function of x

Here P.I. $= \frac{1}{F(D)}(xV)$.

By Leibnitz's theorem, we have

$$D^n (xV) = x \cdot D^n V + nD^{n-1} V = x\, D^n V + \left(\frac{d}{dD} D^n\right) V,$$

showing that $F(D)(xV) = xF(D)V + F'(D)V$. ...(1)

Taking $\frac{1}{F(D)} V$ in place of V in (1) and then operating on both sides by $\frac{1}{F(D)}$, we get after transposition

$$\mathbf{\frac{1}{F(D)}(xV) = x\frac{1}{F(D)}V - \frac{F'(D)}{\{F(D)\}^2}V} \text{ [Remember]}$$

or $$\frac{1}{F(D)}(xV) = x\frac{1}{F(D)}V + \left\{\frac{d}{dD}\frac{1}{F(D)}\right\}V.$$

Proceeding by repeated application of the above formula $\{1/F(D)\}\, x^m V$ can be determined.

Note: The method of article 8 cannot be used when V is of the form $\cos ax$ or $\sin ax$ and $F(-a^2)=0$ *i.e.*, $F(D^2)$ vanishes by putting $-a^2$ for D^2. In that case P.I. is found by the method discussed in article 7.

Illustrative Examples

Example 30: *Solve* $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=x\sin x$.

(Rohilkhand 2010)

Solution: The auxiliary equation is $m^2-2m+1=0$.

$\therefore$ $m=1,1$. Hence C.F. $=(c_1+c_2x)\,e^x$.

And
$$\text{P.I.}=\frac{1}{(D-1)^2}x\sin x=x\frac{1}{(D-1)^2}\sin x-\frac{2}{(D-1)^3}\sin x,\text{ by article 8}$$
$$=x\frac{1}{D^2-2D+1}\sin x-\frac{2}{D^3-3D^2+3D-1}\sin x$$
$$=x\cdot\frac{1}{-2D}\sin x-\frac{2}{-D+3+3D-1}\sin x,\text{ putting }-1^2\ i.e.,\ -1\text{ for }D^2$$
$$=-\frac{x}{2}\cdot\frac{1}{D}\sin x-\frac{2}{2(D+1)}\sin x=\frac{x}{2}\cos x-\frac{1}{D+1}\sin x$$
$$=\frac{x}{2}\cos x-\frac{(D-1)}{(D+1)(D-1)}\sin x=\frac{x\cos x}{2}-\frac{D-1}{D^2-1}\sin x$$
$$=\frac{1}{2}x\cos x+\frac{1}{2}(D-1)\sin x=\frac{1}{2}x\cos x+\frac{1}{2}(\cos x-\sin x).$$

Hence the complete solution is $y=(\text{C.F.})+(\text{P.I.})$

or
$$y=(c_1+c_2x)\,c^x+\frac{1}{2}x\cos x+\frac{1}{2}(\cos x-\sin x).$$

Example 31: *Solve* $\dfrac{d^2y}{dx^2}+4y=x\sin x$.

Solution: The auxiliary equation is $m^2+4=0$.

Therefore $m=0\pm 2i$.

$\therefore$ C.F. $=c_1\cos 2x+c_2\sin 2x$. $[\because e^{0x}=1]$

And
$$\text{P.I.}=\frac{1}{D^2+4}x\sin x=x\frac{1}{D^2+4}\sin x-\frac{2D}{(D^2+4)^2}\sin x,\text{ by article 8}$$
$$=\frac{x\sin x}{-1^2+4}-\frac{2D}{(-1^2+4)^2}\sin x,\text{ putting }-1^2\text{ for }D^2$$
$$=\frac{1}{3}x\sin x-\frac{2}{9}D(\sin x)=\frac{1}{3}x\sin x-\frac{2}{9}\cos x.$$

Hence the complete solution is $y=(\text{C.F.})+(\text{P.I.})$

or $$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} x \sin x - \frac{2}{9} \cos x .$$

Example 32: *Solve* $(D^4 - 1)\, y = x \sin x$. **(Lucknow 2006)**

Solution: The auxiliary equation is $m^4 - 1 = 0$ or $(m^2 - 1)(m^2 + 1) = 0$.

$\therefore$ $m = 1, -1, 0 \pm i$.

$\therefore$ $\text{C.F.} = c_1 e^x + c_2 e^{-x} + e^{0x} (c_3 \cos x + c_4 \sin x)$.

And $\text{P.I.} = \dfrac{1}{D^4 - 1} x \sin x$.

Here we cannot use the method given in 8 because $D^4 - 1$ vanishes by putting -1^2 *i.e.*, -1 for D^2. So here we shall proceed by the method given in article 7.

$\therefore$ $\text{P.I.} = \text{Imaginary part of } \dfrac{1}{D^4 - 1} x e^{ix}$, $[\because e^{ix} = \cos x + i \sin x]$

$= \text{I. P. of } e^{ix} \dfrac{1}{\{(D + i)^4 - 1\}} x$, by using the method for $e^{ax} V$

$$= \text{I. P. of } e^{ix} \frac{1}{(D^4 + 4iD^3 + 6i^2 D^2 + 4i^3 D + i^4 - 1)} x$$

$$= \text{I. P. of } e^{ix} \frac{1}{D^4 + 4iD^3 - 6D^2 - 4iD} x, \qquad [\because i^2 = -1]$$

$$= \text{I. P. of } e^{ix} \frac{1}{-4iD\,[1 + (3D/2i) - D^2 - (D^3/4i)]} x$$

$$= \text{I. P. of } \frac{e^{ix}}{-4\,i} \cdot \frac{1}{D} \left\{1 + \left(\frac{3D}{2i} - \ldots.\right)\right\}^{-1} x$$

$$= \text{I. P. of } \frac{e^{ix}}{-4\,i} \cdot \frac{1}{D} \left\{1 - \frac{3D}{2i} + \ldots\right\} x$$

$$= \text{I. P. of } \frac{e^{ix}}{-4i} \frac{1}{D} \left(x - \frac{3}{2i}\right), \qquad \left[\because D \equiv \frac{d}{dx}\right]$$

$$= \text{I. P. of } \frac{e^{ix}}{-4i} \left(\frac{x^2}{2} - \frac{3x}{2i}\right), \qquad [\because (1/D) \text{ stands for integration w.r.t. } x]$$

$$= \text{I. P. of } \frac{1}{4} i\, e^{ix} \left(\frac{1}{2} x^2 + \frac{3}{2} i\, x\right); \qquad [\because i^2 = -1 \Rightarrow 1/i = -i]$$

$$= \text{I. P. of } \frac{1}{4} i (\cos x + i \sin x) \left(\frac{1}{2} x^2 + \frac{3}{2} ix\right)$$

$$= \frac{1}{4} \left(\frac{1}{2} x^2 \cos x - \frac{3}{2} x \sin x\right)$$

$$= \frac{1}{8} (x^2 \cos x - 3x \sin x) .$$

Hence the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $$y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + \frac{1}{8}(x^2 \cos x - 3x \sin x).$$

Example 33: *Solve* $(D^4 + 2D^2 + 1)\, y = x^2 \cos x$.

(Gorakhpur 2010; Purvanchal 07, 08; Garhwal 09, 11)

Solution: The auxiliary equation is $m^4 + 2m^2 + 1 = 0$

or $(m^2 + 1)^2 = 0$ giving $m = \pm\, i,\ \pm\, i$.

$\therefore$ C. F. $= (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$. $[\because e^{0x} = 1]$

And
$$\text{P. I.} = \frac{1}{D^4 + 2D^2 + 1} x^2 \cos x$$

$$= \text{Real part of } \frac{1}{\{(D^2 + 1)^2} x^2 e^{ix}, \qquad [\because e^{ix} = \cos x + i \sin x]$$

$$= \text{R. P. of } e^{ix} \frac{1}{\{(D + i)^2 + 1\}^2} x^2, \text{by article 5}$$

$$= \text{R. P. of } e^{ix} \frac{1}{(D^2 + 2iD)^2} x^2, \qquad [\because i^2 = -1]$$

$$= \text{R. P. of } e^{ix} \frac{1}{4i^2 D^2 \,[1 + (D/2i)]^2} x^2$$

$$= \text{R. P. of} -\frac{1}{4} e^{ix} \frac{1}{D^2} \left[1 + \frac{D}{2i}\right]^{-2} x^2, \qquad [\because i^2 = -1]$$

$$= \text{R. P. of} -\frac{1}{4} e^{ix} \frac{1}{D^2} [1 - \frac{1}{2} iD]^{-2} x^2, \qquad \left[\because \frac{1}{i} = -i\right]$$

$$= \text{R. P. of} -\frac{1}{4} e^{ix} \frac{1}{D^2} \left[1 + 2 \cdot \frac{1}{2} iD + 3 \cdot \frac{1}{4} i^2 D^2 + \ldots\right] x^2,$$

expanding by binomial theorem

$$= \text{R. P. of} -\frac{1}{4} e^{ix} \frac{1}{D^2} \left[1 + iD - \frac{3}{4} D^2 + \ldots\right] x^2$$

$$= \text{R. P. of} -\frac{1}{4} e^{ix} \left[\frac{1}{D^2} + \frac{i}{D} - \frac{3}{4} + \text{terms in } D, D^2, \text{and so on}\right] x^2$$

$$= \text{R. P. of} -\frac{1}{4} e^{ix} \left[\frac{1}{3} \frac{x^4}{4} + i \frac{1}{3} x^3 - \frac{3}{4} x^2 + \text{terms in } x^1, x^0\right],$$

($\because 1/D$ stands for integration w.r.t. x)

$$= \text{R. P. of} -\frac{1}{4} (\cos x + i \sin x) \left(\frac{1}{12} x^4 + \frac{1}{3} ix^3 - \frac{3}{4} x^2 + \text{terms in } x^1, x^0\right)$$

$$= -\frac{1}{4} \left(\frac{1}{12} x^4 - \frac{3}{4} x^2\right) \cos x + \frac{1}{4} \left(\frac{1}{3} x^3\right) \sin x$$

+ terms already included in the C.F.

$$= -\frac{1}{48}(x^4 - 9x^2)\cos x + \frac{1}{12}x^3 \cdot \sin x\,,$$

neglecting the terms already included in the C.F.

Hence the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $$y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x - \frac{1}{48}(x^4 - 9x^2)\cos x + \frac{1}{12}x^3 \sin x\,.$$

Example 34: *Solve* $(D^2 - 2D + 1)\, y = xe^x \sin x$.

(Purvanchal 2010; Kanpur 12; Avadh 13; Kumaun 13)

Solution: The auxiliary equation is

$$m^2 - 2m + 1 = 0 \quad \text{or} \quad (m-1)^2 = 0\,. \quad \therefore \quad m = 1, 1\,.$$

$\therefore$ $$\text{C.F.} = (c_1 + c_2 x)\, e^x.$$

And $$\text{P.I.} = \frac{1}{D^2 - 2D + 1}\, xe^x \sin x = \frac{1}{(D-1)^2}\, e^x\, x \sin x$$

$$= e^x \frac{1}{\{(D+1) - 1\}^2}\, x \sin x\,, \text{ by 4.5}$$

$$= e^x \frac{1}{D^2}(x \sin x) = e^x \cdot \frac{1}{D}\int x \sin x\, dx,$$

[$\because$ $(1/D)$ stands for integration w.r.t. x]

$$= e^x \frac{1}{D}(-x\cos x + \sin x)\,, \text{integrating by parts}$$

$$= e^x \int (-x \cos x + \sin x)\, dx = e^x \left[\int -x\cos x\, dx + \int \sin x\, dx\right]$$

$$= e^x \left[-x \sin x + \int \sin x\, dx + \int \sin x\, dx\right] = e^x(-x \sin x - 2\cos x)\,.$$

Hence the complete solution is $y = (\text{C.F.}) + (\text{P.I.})$

or $$y = (c_1 + c_2 x)\, e^x - e^x (x \sin x + 2\cos x)\,.$$

Comprehensive Exercise 8

Solve the following differential equations :

1. $d^2 y/dx^2 + 9y = x \sin x$.
2. $(D^2 + 2D + 1)\, y = x \cos x$.
3. $(D^2 - 4D + 4)\, y = 8x^2 e^{2x} \sin 2x$. **(Gorakhpur 2011; Meerut 05B; Garhwal 12)**
4. $(D^2 + 4)\, y = x \sin 2x$.
5. $(D^2 + D)\, y = x \cos x$.
6. $(D^2 - 1)\, y = x^2 \cos x$.
7. $(D^2 + 1)\, y = x^2 \sin 2x$. **(Meerut 2004, 07B; Lucknow 11)**

8. $(D^4 - 1)\ y = x \sin 2x$. **(Meerut 2001, 06B)**

9. $(D^2 + m^2)\ y = x \cos mx$.

10. $(d^2 y/dx^2) - y = x \sin x + (1 + x^2)\ e^x$. **(Bundelkhand 2009)**

Answers 8

1. $y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{8} x \sin x - \frac{1}{32} \cos x$
2. $y = (c_1 + c_2 x)\ e^{-x} + \frac{1}{2} x \sin x - \frac{1}{2} \sin x + \frac{1}{2} \cos x$
3. $y = (c_1 + c_2 x)\ e^{2x} + e^{2x}\ (-2x^2 \sin 2x + 3 \sin 2x - 4x \cos x)$
4. $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{8} x^2 \cos 2x + \frac{1}{16} x \sin 2x$
5. $y = c_1 + c_2\ e^{-x} - \frac{1}{2} x\ (\cos x - \sin x) + \cos x + \frac{1}{2} \sin x$
6. $y = c_1 e^x + c_2 e^{-x} - \frac{1}{2} (x^2 - 1) \cos x + x \sin x$
7. $y = c_1 \cos x + c_2 \sin x - \frac{1}{27} [24x \cos 2x - (9x^2 - 26) \sin 2x]$
8. $y = c_1\ e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + \frac{1}{15} \left(\frac{32}{15} \cos 2x + x \sin 2x \right)$
9. $y = c_1 \cos mx + c_2 \sin mx + (x/4m^2) \cos mx + (x^2/4m) \sin mx$
10. $y = c_1 e^x + c_2 e^{-x} - \frac{1}{2} (\cos x + x \sin x) + \frac{1}{12} e^x\ (2x^3 - 3x^2 + 9x)$

9 The Operator $\frac{1}{D-\alpha}$ and $\frac{1}{D+\alpha}$, α being a Constant

If Q is any function of x, then

$$\frac{1}{D-\alpha} Q = e^{\alpha x} \int e^{-\alpha x} Q\, dx \text{ and } \frac{1}{D+\alpha} Q = e^{-\alpha x} \int e^{\alpha x} Q\, dx$$

(Meerut 20, 03; Rohilkhand 09)

Proof. We have $\frac{1}{D-\alpha} Q = \frac{1}{D-\alpha} [e^{\alpha x} (e^{-\alpha x} Q)]$

$$= e^{\alpha x} \frac{1}{(D+\alpha)-\alpha} e^{-\alpha x} Q, \text{ by article 5}$$

$$= e^{\alpha x} \frac{1}{D} e^{-\alpha x} Q = e^{\alpha x} \int e^{-\alpha x} Q\, dx.$$

Similarly proceed for second operator proof.

Illustrative Examples

Example 35: *Solve* $\frac{d^2 y}{dx^2} + a^2 y = sec\ ax$.

(Garhwal 2002; Agra 06; Bundelkhand 07; Purvanchal 07, 08; Lucknow 09)

Solution: The auxiliary equation is

$$m^2 + a^2 = 0 \quad \text{or} \quad m = \pm ia.$$

$\therefore$ $$\text{C.F.} = c_1 \cos ax + c_2 \sin ax.$$

And $$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax = \frac{1}{(D + ia)(D - ia)} \sec ax$$

$$= \frac{1}{2\,ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right] \sec ax \text{ [by resolving into partial fractions]}$$

$$= \frac{1}{2\,ia}\left[\frac{1}{D - ia} \sec ax - \frac{1}{D + ia} \sec ax\right]$$

$$= \frac{1}{2\,ia}\left[e^{iax} \int e^{-iax} \sec ax\, dx - e^{-iax} \int e^{iax} \sec ax\, dx\right], \text{ by article 9}$$

$$= \frac{1}{2\,ia}\left[e^{iax} \int \frac{\cos ax - i \sin ax}{\cos ax} dx - e^{-iax} \int \frac{\cos ax + i \sin ax}{\cos ax} dx\right],$$

$$(\because\ e^{-iax} = \cos ax - i \sin ax \text{ and } e^{iax} = \cos ax + i \sin ax)$$

$$= \frac{1}{2\,ia}\left[e^{iax}\left\{x + \frac{i}{a} \log \cos ax\right\} - e^{-iax}\left\{x - \frac{i}{a} \log \cos ax\right\}\right]$$

$$= \frac{x}{a}\left\{\frac{e^{iax} - e^{-iax}}{2i}\right\} + \frac{1}{a^2} (\log \cos ax)\left\{\frac{e^{iax} + e^{-iax}}{2}\right\}$$

$$= \frac{x}{a} \sin ax + \frac{1}{a^2} (\log \cos ax) \cos ax.$$

Hence the complete solution is

$$y = (\text{C.F.}) + (\text{P.I.})$$

or $$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax \cdot \log(\cos ax).$$

Example 36: *Solve* $\frac{d^2 y}{dx^2} + 4y = 4tan2x.$ **(Garhwal 2015)**

Solution: The auxiliary equation is

$$m^2 + 4 = 0$$

or $$m = \pm 2i.$$

$\therefore$ $$\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x.$$

And $$\text{P.I.} = \frac{1}{D^2+2^2}4\tan 2x = \frac{1}{(D+2i)(D-2i)}4\tan 2x$$

$$= \frac{1}{i}\left[\frac{1}{D-2i} - \frac{1}{D+2i}\right]\tan 2x = \frac{1}{i}\left[\frac{1}{D-2i}\tan 2x - \frac{1}{D+2i}\tan 2x\right]$$

Now $$\frac{1}{D-2i}\tan 2x = e^{2ix}\int e^{-2ix}\tan 2x\,dx = e^{2ix}\int(\cos 2x - i\sin 2x)\frac{\sin 2x}{\cos 2x}dx$$

$$= e^{2ix}\int\left(\sin 2x - \frac{i\sin^2 2x}{\cos 2x}\right)dx = e^{2ix}\int\left(\sin 2x - \frac{i(1-\cos^2 2x)}{\cos 2x}\right)dx$$

$$= e^{2ix}\int\{\sin 2x - i\sec 2x + i\cos 2x\}\,dx = e^{2ix}i\int(e^{-2ix} - \sec 2x)\,dx$$

$$= ie^{2ix}\left[\frac{e^{-2ix}}{-2i} - \frac{1}{2}\log(\sec 2x + \tan 2x)\right]$$

$$= -\frac{1}{2} - \frac{ie^{2ix}}{2}\log(\sec 2x + \tan 2x).$$

Similarly, $$\frac{1}{D+2i}\tan 2x = \frac{-1}{2} + \frac{ie^{2ix}}{2}\log(\sec 2x + \tan 2x)$$

$$\text{P.I.} = \frac{1}{i}\left[-i\left(\frac{e^{2ix}+e^{-2ix}}{2}\right)\log(\sec 2x + \tan 2x)\right]$$

$$= -\cos 2x\log(\sec 2x + \tan 2x).$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1\cos 2x + c_2\sin 2x - \cos 2x\log(\sec 2x + \tan 2x).$$

Comprehensive Exercise 9

Solve the following differential equations :

1. $(d^2y/dx^2) + 9y = \sec 3x.$ **(Meerut 2007)**
2. $(D^2 + a^2)\,y = \operatorname{cosec} ax.$ **(Lucknow 2008)**
3. $(D^2 + a^2)\,y = \tan ax.$
4. $(D^2 + 3D + 2)\,y = e^{e^x}.$ **(Kumaun 2014)**
5. $(D^2 + 1)\,y = x - \cot x.$

Answers 9

1. $y = c_1\cos 3x + c_2\sin 3x + \frac{1}{3}\,x\sin 3x + \frac{1}{9}\cos 3x\log(\cos 3x)$

2. $y = c_1 \cos ax + c_2 \sin ax + (1/a^2) \sin ax \log(\sin ax) - (x/a) \cos ax$

3. $y = c_1 \cos ax + c_2 \sin ax - (1/a^2) \cos ax \log \tan\left(\frac{1}{4}\pi + \frac{1}{2}ax\right)$

4. $y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$

5. $y = c_1 \cos x + c_2 \sin x + x - \sin x \log(\text{cosec } x - \cot x)$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The solution of differential equation $\dfrac{d^2y}{dx^2} + y = 0$ is

   (a) $c_1 e^{-x} + c_2 e^x$
   (b) $c_1 \cos x + c_2 \sin x$
   (c) $(c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$
   (d) none of these.

(Bundelkhand 2001; Kumaun 07)

2. P.I. of the differential equation $(D^2 + D + 1)\, y = e^x$ is

   (a) $\dfrac{1}{3} e^x$
   (b) $3e^x$
   (c) e^x
   (d) none of these.

(Avadh 2005; Garhwal 11)

3. For the differential equation $F(D)\, y = e^{ax}$, if $F(a) = 0$, then P.I. $= \dfrac{1}{F(D)} e^{ax}$ is given by

   (a) $\dfrac{1}{F(a)} e^{ax}$
   (b) $\dfrac{1}{F(-a)} e^{ax}$
   (c) $e^{ax} \dfrac{1}{F(D + a)} 1$
   (d) none of these

(Rohilkhand 2005; Kumaun 11)

4. For the particular integral $\dfrac{1}{f(D^2)} \sin ax$ when $f(-a^2) = 0$, which one of the following is correct

   (a) $\dfrac{1}{f(D^2)} \sin ax = \dfrac{1}{f(-a^2)} \sin ax$
   (b) $\dfrac{1}{D^2 + a^2} \sin ax = -\dfrac{x}{2a} \cos ax$
   (c) $\dfrac{1}{D^2 + a^2} \sin ax = \dfrac{x}{2a} \cos ax$
   (d) none of these

(Bundelkhand 2001; Agra 08; Kumaun 12)

5. The solution of the differential equation $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$ is

(a) $y = c_1 e^x + c_2 e^{3x} + x$ (b) $y = (c_1 + c_2)\, e^x - xe^x$

(c) $y = c_1\, e^x + c_2 e^{2x} - xe^x$ (d) $y = c_1\, x + c_2\, e^x - e^{2x}$.

6. The P.I. of the differential equation $(D^2 + 1)\, y = \cos x$ is

(a) $\frac{1}{2}\sin x$ (b) $-\frac{x}{2}\sin x$

(c) $\frac{x}{2}\sin x$ (d) $\frac{x}{2}\cos x$ **(Rohilkhand 2007)**

7. The solution of differential equation $\frac{d^2 y}{dx^2} + a^2 y = 0$ is

(a) $y = c_1 e^{ax} + c_2 e^{-ax}$ (b) $y = c_1 \sin ax + c_2 \cos ax$

(c) $y = \sin ax\, e^{ax}$ (d) $y = \cos ax\, e^{ax}$

(Garhwal 2005, 08; Kumaun 10)

8. The solution of the differential equation $(D^2 + 9)\,y = 0$ is

(a) $y = c_1 e^{3x} + c_2 e^{-3x}$ (b) $y = (c_1 + c_2 x)\, e^{3x}$

(c) $y = c_1 \cos 3x + c_2 \sin 3x$ (d) $y = (c_1 + c_2 x) \sin 3x$

(Garhwal 2006)

9. The general solution of $(D^2 - m^2)\,y = 0$ is

(a) $y = (c_1 + c_2 x)\, e^{mx}$ (b) $y = c_1 \sin mx + c_2 \cos mx$

(c) $y = c_1 e^{mx} + c_2 e^{-mx}$ (d) none of these **(Garhwal 2007)**

10. The P.I. of the differential equation $(D^2 - 3D + 2)\,y = e^{5x}$ is

(a) $\frac{1}{2} e^{5x}$ (b) $\frac{1}{5} e^{-5x}$

(c) $\frac{1}{12} e^{5x}$ (d) 0 **(Garhwal 2010B)**

11. C.F. of the differential equation $(D^2 - 3D + 2)\, y = e^x$ is

(a) $c_1 e^{-x} + c_2 e^{-2x}$ (b) $c_1 e^x + c_2 e^{-2x}$

(c) $c_1 e^{-x} + c_2 e^{2x}$ (d) $c_1 e^x + c_2 e^{2x}$ **(Garhwal 2013)**

12. A part of the C.F. of the differential equation $\frac{d^2 y}{dx^2} + a^2 y = \cos ax$ is

(a) $c_1 \sin ax - c_2 \cos ax$ (b) $-(c_1 \sin ax + c_2 \cos ax)$

(c) $c_1 \cos ax + c_2 \sin ax$ (d) $-c_1 \sin^{-1} ax + c_2 \cos^{-1} ax$

(Garhwal 2015)

13. The solution of the differential equation $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 0$ is

(a) $y = c_1e^{2x} + c_2e^{-2x}$
(b) $y = c_1e^{x} + c_2e^{2x}$
(c) $y = c_1x + c_2e^{-2x}$
(d) $y = (c_1 + c_2x)e^{2x} + c_3e^{-2x}$

14. The particular integral of the differential equation $y'' - 5y' + 4y = 0$ is

(a) 0
(b) $y = 4e^{x} + e^{4x}$
(c) $y = c_1e^{4x} + c_2e^{x}$
(d) $y = 2x$

(Kumaun 2014)

15. For the differential equation $F(D)y = e^{ax}$ if $F(a) \neq 0$, then the value of the particular integral $\frac{1}{F(D)}e^{ax}$ is

(a) $-\frac{e^{ax}}{F(a)}$
(b) $\frac{e^{-ax}}{F(a)}$
(c) $\frac{e^{ax}}{F(a)}$
(d) none of these

(Kumaun 2013)

16. The value of particular integral $\frac{1}{D^2 + a^2}\sin ax$ of the differential equation $(D^2 + a^2)y = \sin ax$ is

(a) $x\int \sin ax\, dx$
(b) $\frac{x}{2}\int \sin ax\, dx$
(c) $\frac{x}{2}\int \cos ax\, dx$
(d) none of these

(Kumaun 2013)

17. The general solution of the differential equation $y'' - 2y' + 5y = 0$ will be

(a) $y = e^{x}(c_1\cos 2x + c_2\sin 3x)$
(b) $y = c_1e^{x} + c_2e^{3x}$
(c) $y = e^{x}(c_1\cos 2x + c_2\sin 2x)$
(d) $y = c_1\cos 2x + c_2\sin 3x$

(Kumaun 2014)

18. The particular integral of $(D^2 + 4)y = \cos 2x$ is

(a) $\frac{x}{4}\cos 2x$
(b) $\frac{x}{4}\sin 2x$
(c) $x\cos 2x$
(d) $\frac{1}{4}\sin 2x$

(Kanpur 2016)

19. The solution of differential equation $\frac{d^2y}{dx^2} + y = 0$ satisfying the initial condition $y(0) = 1, y(\pi/2) = 2$ is

(a) $y = 2\cos x + \sin x$
(b) $y = \cos x + 2\sin x$
(c) $y = \cos x + \sin x$
(d) $y = 2\cos x + 2\sin x$

(Kanpur 2016)

20. Particular integral of $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^2$ is

(a) $x^2 - 2x$
(b) $x^2 + 2x$
(c) $x^2 + 2$
(d) $x^2 - 2$

(Kanpur 2016)

21. $\dfrac{1}{f(D)} x^n \cos ax$ is equal to

(a) real part of $e^{iax} \dfrac{1}{f(D+ia)} x^n$

(b) imaginary part of $e^{iax} \dfrac{1}{f(D+ia)} x^n$

(c) $e^{iax} \dfrac{1}{f(D+ia)} x^n$

(d) none of these **(Kanpur 2016)**

22. The solution of $(D^2+2D+2)y=0,\ y(0)=0,\ y'(0)=1$ is

(a) $e^x \sin x$ (b) $e^{-x} \cos x$

(c) $e^{-x} \sin x$ (d) $e^x \cos x$ **(Kanpur 2016)**

23. The roots of the auxiliary equation of the differential equation $\dfrac{d^2y}{dt^2} - 6\dfrac{dy}{dt} + 9y = 4e^{3t}$ are

(a) 3, – 3 (b) 3, 3

(c) –3, – 3 (d) none of these **(Kanpur 2016)**

24. Complete solution of the differential equation $y'' + y = \sec ax$ is

(a) $y = c_1 \cos x + c_2 \sin x + \cos x \log \cos x + x \sin x$

(b) $y = c_1 \cos x + c_2 \sin x + x \sin x$

(c) $y = c_1 \cos x - c_2 \sin x$

(d) $y = c_1 \cos x + c_2 \sin x + \dfrac{x^2}{2}$ **(Kanpur 2016)**

25. Particular integral of $\dfrac{d^2y}{dx^2} + 9y = \sin 3x$ is

(a) $-\dfrac{x}{6}\cos 3x$ (b) $\dfrac{x}{6}\cos 3x$

(c) $\dfrac{x}{6}\sin 3x$ (d) $-\dfrac{x}{6}\sin 3x$ **(Kanpur 2016)**

26. Solution of $(D^3+2D^2+D)y=0$ is

(a) $c_1 e^{-x} + c_2$ (b) $(c_1 + c_2 x)e^{-x}$

(c) $(c_1 + c_2 x + c_3 x^2)e^{-x}$ (d) $c_1 + (c_2 + c_3 x)e^{-x}$

27. Complementary function of $(D^4 + 2D^3 - 3D)y = x^3$ is

(a) $(c_1 + c_2 x) + c_3 e^x + c_4 e^{-x}$ (b) $(c_1 + c_2 x) + c_3 e^x + c_4 e^{-3x}$

(c) $(c_1 + c_2 x + c_3 x)e^{-x} + c_4 e^{-3x}$ (d) $(c_1 + c_2 x)e^{-3x} + (c_3 + c_4 x)e^{-x}$

28. Complementary function of $(D^2 - 4)y = 0$ is

(a) $(c_1 + c_2 x)e^{2x}$ (b) $(c_1 + c_2 x)e^{-2x}$

(c) $c_1 e^{2x} + c_2 e^{-2x}$ (d) $c_1 \sin 2x + c_2 \cos 2x$

29. Complementary function of $(D^2+1)y=0$ is

(a) c_1+c_2x (b) $(c_1+c_2x)e^{-x}$

(c) $(c_1+c_2x)e^{x}$ (d) $c_1\cos x+c_2\sin x$

30. The roots of the auxiliary equation of $(D^4+2D^2+1)y=0$ are

(a) $m=\pm 1, \pm 1$ (b) $m=\pm 1, \pm i$

(c) $m=\pm i, \pm i$ (d) $m=1, 1, i, i$

31. The roots of the auxiliary equation of $(D^4-1)y=0$ are

(a) $m=1, -1, i, -i$ (b) $m=\pm 1, \pm 1$

(c) $m=\pm i, \pm i$ (d) $m=1, 1, \pm i$

32. The roots of the auxiliary equation of $(D^2-2D+5)y=0$ are

(a) $1+2i$ (b) $1-2i$

(c) $1\pm 2i$ (d) $2\pm i$

33. Complementary function of $(D^4+2a^2D^2+a^4)y=\cos ax$ is

(a) $(c_1+c_2)x\cos ax+(c_3+c_4)x\sin ax$

(b) $(c_1+c_2x)\cos ax+(c_3+c_4x)\sin ax$

(c) $(c_1x+c_2x^2)\cos ax+(c_3x+c_4x^2)\sin ax$

(d) none of these

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. The general solution of the equation $f(D)\,y=0$ is called the ...... of the equation $f(D)\,y=Q$.

2. The solution of the differential equation $\dfrac{d^2y}{dx^2}+a^2y=0$ is ....

3. The complementary function (C.F.) of the differential equation

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+4y=0 \text{ is ....}$$

4. The particular integral (P.I.) of the differential equation

$$\frac{d^3y}{dx^3}-4\frac{d^2y}{dx^2}+5\frac{dy}{dx}-2=0 \text{ is ....}$$

5. The particular integral of the differential equation $f(D)\,y=Q$ will be ....

6. The particular integral of the differential equation

$$\frac{d^2y}{dx^2}+a^2y=\sin ax \text{ is ....}$$

(Kumaun 2009)

7. For the differential equation $F(D)\,y=e^{-ax}$ if $F(-a)=0$ then P.I. $=\dfrac{1}{F(D)}e^{-ax}$ is ....

(Kumaun 2009)

True or False

Write 'T' for true and 'F' for false statement.

1. A linear differential equation is an equation in which the dependent variable and its derivatives appear only in the first degree.
2. $\frac{d^4 y}{dx^4} - 5\frac{d^3 y}{dx^3} - 2\frac{dy}{dx} + 4xy = 0$ is a linear differential equation with constant coefficients. **(Meerut 2003)**
3. $e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + c_3 e^{2x}$ is the solution of the differential equation $\frac{d^3 y}{dx^3} - 8y = 0$.

Answers

Multiple Choice Questions

1. (b) 2. (a) 3. (c) 4. (b) 5. (c)
6. (c) 7. (b) 8. (c) 9. (c) 10. (c)
11. (d) 12. (c) 13. (d) 14. (a) 15. (c)
16. (b) 17. (c) 18. (b) 19. (b) 20. (a)
21. (a) 22. (c) 23. (b) 24. (a) 25. (a)
26. (d) 27. (b) 28. (c) 29. (d) 30. (c)
31. (a) 32. (c) 33. (b)

Fill in the Blank(s)

1. C.F. 2. $y = c_1 \cos ax + c_2 \sin ax$ 3. $c_1 e^{x} + c_2 e^{4x}$
4. $\frac{2}{5}x$ 5. $\frac{1}{F(D)}Q$ 6. $\frac{-x}{2a}\cos ax$ 7. $\frac{e^{-ax}}{F(D-a)}\cdot 1$

True or False

1. *T* 2. *F* 3. *T*

Chapter

5

Homogeneous Linear Differential Equations (Cauchy-Euler Equations)

1 Homogeneous Linear Differential Equations

(Meerut 2009B)

Definition: A differential equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}}{dx^{n-1}} + \ldots + a_{n-1} x \frac{dy}{dx} + a_n y = Q, \qquad \ldots(1)$$

where $a_1, a_2, \ldots, a_n$ are constants and Q is either a constant or a function of x is called a **homogeneous** linear differential equation of order n.

A linear homogeneous differential equation with variable coefficients is called **Cauchy-Euler equation** or **Euler-Cauchy equation** or simply **Euler's equation**.

2 Method of Solution

To solve such equations, we introduce a new independent variable z such that

$x = e^z$ or $\log x = z$, so that $1/x = dz/dx$.

We have $\qquad \dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx} = \dfrac{1}{x} \dfrac{dy}{dz} \cdot$

$\therefore$ $$x\frac{dy}{dx}=\frac{dy}{dz} \text{ so that } x\frac{d}{dx}\equiv\frac{d}{dz}\equiv D, \text{ say.}$$

Now $$x\frac{d}{dx}\left(x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}\right)=x^n\frac{d^n y}{dx^n}+(n-1)x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}$$

or $$x^n\frac{d^n y}{dx^n}=\left(x\frac{d}{dx}-n+1\right)x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}$$

$$=(D-n+1)\,x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}. \qquad \ldots(2)$$

Putting $n=2,3,4,\ldots$ etc. in (2), we have

$$x^2\frac{d^2y}{dx^2}=(D-1)\,x\frac{dy}{dx}=(D-1)\,Dy \qquad \left[\because x\frac{d}{dx}\equiv D\right]$$

$$=D(D-1)\,y, \text{ because the operators can be interchanged}$$

$$x^3\frac{d^3y}{dx^3}=(D-2)\,x^2\frac{d^2y}{dx^2}$$

$$=(D-2)(D-1)\,Dy$$

$$=D(D-1)(D-2)\,y.$$

Whence, generalizing, we have

$$\boldsymbol{x^n\frac{d^n y}{dx^n}=D(D-1)(D-2)\ldots(D-n+1)\,y.}$$

Substituting these values of $x\frac{dy}{dx}, x^2\frac{d^2y}{dx^2},\ldots,x^n\frac{d^ny}{dx^n}$ in (1) and thus changing the independent variable from x to z, we have

$$[\{D(D-1)\ldots(D-n+1)\}+a_1\{D(D-1)\ldots(D-n+2)\}$$
$$+\ldots+a_{n-1}D+a_n]\,y=Q,$$

or $$f(D)\,y=Q, \qquad \ldots(3)$$

where Q has now become a function of z.

In the differential equation (3) the independent variable is z and the operator D stands for d/dz. Obviously (3) is a linear differential equation with constant coefficients and so it can be solved by the methods given in the previous chapter *i.e.*, chapter 3. Thus as in the case of the linear equations with constant coefficients, the general solution of (3) is the sum of any **particular integral** of (3) and the **complementary function** *i.e.*, the general solution of

$$f(D)\,y=0. \qquad \ldots(4)$$

To find the Complementary function (C.F.):

(i) If $m_1, m_2, \ldots, m_n$ are the roots of the **auxiliary equation** of (4), and no two of them are equal, the general solution of (4) *i.e.*, the C.F. of the solution of (3) is easily seen to be

$$y = c_1 e^{m_1 z} + c_2 e^{m_2 z} + \ldots + c_n e^{m_n z},$$

or $$y = c_1 x^{m_1} + c_2 x^{m_2} + \ldots + c_n x^{m_n}. \qquad [\because e^z = x]$$

(ii) In case there are r roots alike, each equal to m, and the rest all different, then the

$$\text{C.F.} = (c_1 + c_2 z + \ldots + c_r z^{r-1}) e^{mz} + c_{r+1} e^{m_{r+1} z} + \ldots + c_n e^{m_n z}$$

$$= [c_1 + c_2 \log x + \ldots + c_r (\log x)^{r-1}] x^m + c_{r+1} x^{m_{r+1}} + \ldots + c_n x^{m_n}.$$

(iii) In case the roots are imaginary, say of the form $\alpha \pm i\beta$, then the

$$\text{C.F.} = e^{\alpha x}(c_1 \cos \beta z + c_2 \sin \beta z) = x^{\alpha} [c_1 \cos (\beta \log x) + c_2 \sin (\beta \log x)].$$

In this case we can also write the

$$\text{C.F.} = c_1 e^{\alpha z} \cos (\beta z + c_2) = c_1 x^{\alpha} \cos (\beta \log x + c_2).$$

(iv) In case the roots $\alpha \pm i\beta$ occur r times, the C.F. corresponding to these roots will be

$$e^{\alpha z}[(c_1 + c_2 z + \ldots + c_r z^{r-1}) \cos \beta z + (c_1' + c_2' z + \ldots + c_r' z^{r-1}) \sin \beta z]$$

$$= x^{\alpha} [\{c_1 + c_2 \log x + \ldots + c_r (\log x)^{r-1}\} \cos (\beta \log x)$$

$$+ \{c_1' + c_2' \log x + \ldots + c_r' (\log x)^{r-1}\} \sin (\beta \log x)].$$

To find the Particular integral. (P.I.):

The particular integral (P.I.) of (3) is given by

$$\frac{1}{f(D)} Q.$$

If α is a constant, we have

$$\frac{1}{D-\alpha} Q = \frac{1}{D-\alpha} \{e^{\alpha z}(e^{-\alpha z} Q)\}$$

$$= e^{\alpha z} \frac{1}{(D+\alpha)-\alpha} e^{-\alpha z} Q$$

$$= e^{\alpha z} \frac{1}{D} e^{-\alpha z} Q = e^{\alpha z} \int e^{-\alpha z} Q \, dz.$$

Methods to find the P.I:

General Methods:

(i) We resolve the operator $f(D)$ into linear factors. Thus we write

$f(D) = (D - m_1)(D - m_2) \ldots (D - m_n)$. Then the

$$\text{P.I.} = \frac{1}{f(D)} Q = \frac{1}{(D-m_1)(D-m_2)..(D-m_{n-1})(D-m_n)} Q$$

$$= \frac{1}{(D-m_1)(D-m_2)\ldots(D-m_{n-1})} e^{m_n z} \int e^{-m_n z} Q \, dz,$$

by operating $\frac{1}{D - m_n}$ upon Q as explained above.

Similarly we operate with other remaining factors in succession and thus we find the P.I.

(ii) First we resolve $f(D)$ into linear factors as in (i) and then we break up $\{f(D)\}^{-1}$ into partial fractions. Then the

$$\text{P.I.} = \frac{1}{f(D)} Q = \left\{ \frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n} \right\} Q$$

$$= A_1 e^{m_1 z} \int e^{-m_1 z} Q\, dz + \ldots + A_n e^{m_n z} \int e^{-m_n z} Q\, dz\ .$$

Special short Methods.

(i) When Q is of the form e^{az}, then

$$\text{P.I.} = \frac{1}{f(D)} e^{az} = \frac{1}{f(a)} e^{az}, \text{ provided } f(a) \neq 0\ .$$

(ii) When Q is of the form $\cos az$ or $\sin az$, then the P.I. is given by

$$\frac{1}{f(D^2)} \cos az = \frac{1}{f(-a^2)} \cos az,$$

and $$\frac{1}{f(D^2)} \sin az = \frac{1}{f(-a^2)} \sin az, \qquad [\text{Provided } f(-a^2) \neq 0]$$

(iii) If Q is of the form z^m, we have P.I. $= \dfrac{1}{f(D)} z^m$.

We expand $\{f(D)\}^{-1}$ in ascending powers of D retaining terms as far as D^m and operate each term on z^m.

(iv) If Q is of the form $e^{az} V$, where V is any function of z, we have the

$$\text{P.I.} = \frac{1}{f(D)} e^{az} V = e^{az} \frac{1}{f(D + a)} V.$$

(v) If Q is of the form zV, where V is any function of z, we have the

$$\text{P.I.} = \frac{1}{f(D)} (zV) = z \frac{1}{f(D)} V + \left\{ \frac{d}{dD} \frac{1}{f(D)} \right\} V.$$

Illustrative Examples

Example 1: *Solve* $x^2 D^2 y + 5x\, Dy + 4y = 0,$ *where* $D \equiv d/dx.$ **(Kumaun 2008)**

Solution: The given differential equation is

$$x^2 \frac{d^2 y}{dx^2} + 5x \frac{dy}{dx} + 4y = 0\ . \qquad \ldots(1)$$

Now (1) is a homogeneous linear differential equation of order 2. So putting $x = e^z$ and denoting d/dz by D', we have

$$x \frac{dy}{dx} = D' y, x^2 \frac{d^2 y}{dx^2} = D'(D' - 1)\, y\ .$$

$\therefore$ the differential equation (1) transforms to

$$\{D'(D'-1)+5D'+4\}\ y=0$$

or $$(D'^2+4D'+4)\ y=0$$

or $$(D'+2)^2\ y=0,$$

which is a linear differential equation with constant coefficients, the independent variable being z.

$\therefore$ auxiliary equation is $(m+2)^2=0$, giving $m=-2,-2$.

$\therefore$ the complete solution is

$$y=(c_1+c_2z)\,e^{-2z}$$

or $$y=(c_1+c_2z)\,(e^z)^{-2}$$

or $$y=(c_1+c_2\log x)\,x^{-2}.$$

Example 2: *Solve* $(x^3D^3+3x^2D^2-2xD+2)\ y=0$, *where* $D\equiv d/dx$.

Solution: Putting $x=e^z$ and denoting d/dz by D', we have

$$x\,Dy=x\frac{dy}{dx}=D'\,y,\ x^2D^2y=x^2\frac{d^2y}{dx^2}=D'(D'-1)\ y$$

and $$x^3\ D^3y=x^3\frac{d^3y}{dx^3}=D'(D'-1)(D'-2)\ y.$$

$\therefore$ the given differential equation transforms to

$$[D'(D'-1)(D'-2)+3D'(D'-1)-2D'+2]y=0$$

or $$[D'(D'-1)(D'-2)+3D'(D'-1)-2(D'-1)]\ y=0$$

or $$(D'-1)[D'(D'-2)+3D'-2]\ y=0$$

or $$(D'-1)(D'^2+D'-2)\ y=0$$

or $$(D'-1)(D'+2)(D'-1)\ y=0$$

or $$(D'-1)^2\ (D'+2)\ y=0\ .$$

$\therefore$ the auxiliary equation is $(m-1)^2\ (m+2)=0$, giving

$$m=1,1,-2\ .$$

$\therefore$ the general solution is

$$y=(c_1+c_2z)\,e^z+c_3\,e^{-2z}=(c_1+c_2z)\,e^z+c_3\,(e^z)^{-2}$$

or $$y=(c_1+c_2\log x)\,x+c_3\,x^{-2},\qquad [\because\ x=e^z \text{ and } z=\log x]$$

or $$y=(c_1+c_2\log x)\,x+c_3\cdot(1/x^2).$$

Example 3: *Solve* $x^4\dfrac{d^3y}{dx^3}+2x^3\dfrac{d^2y}{dx^2}-x^2\dfrac{dy}{dx}+xy=1$.

(Garhwal 2011)

Solution: Dividing out by x, the given equation may be written as

$$x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = \frac{1}{x}, \qquad \ldots(1)$$

which is a linear homogeneous equation of order 3.

Putting $x = e^z$ in (1) and denoting d/dz by D, we have

$$[D(D-1)(D-2) + 2D(D-1) - D + 1]\, y = e^{-z}$$

or $$(D-1)^2 (D+1)\, y = e^{-z}.$$

$\therefore$ auxiliary equation is

$$(m-1)^2 (m+1) = 0, \text{ giving } m = 1, 1, -1.$$

$\therefore$ $$\text{C.F.} = (c_1 + c_2 z)\, e^z + c_3\, e^{-z}$$

$$= (c_1 + c_2 \log x)\, x + c_3 x^{-1}. \qquad [\because e^z = x \text{ and } z = \log x]$$

And $$\text{P.I.} = \frac{1}{(D-1)^2 (D+1)}\, e^{-z}$$

$$= \frac{1}{(D+1)} \left[\frac{1}{(D-1)^2}\, e^{-z} \right]$$

$$= \frac{1}{(D+1)} \cdot \frac{1}{4}\, e^{-z}$$

$$= \frac{1}{4} \frac{1}{D+1}\, e^{-z} \cdot 1$$

$$= \frac{1}{4}\, e^{-z} \frac{1}{D-1+1}\, 1$$

$$= \frac{1}{4}\, e^{-z} \frac{1}{D}\, 1$$

$$= \frac{1}{4}\, z e^{-z} = \frac{1}{4x} \log x.$$

Hence the complete solution is

$$y = (c_1 + c_2 \log x)\, x + \frac{c_3}{x} + \frac{1}{4x} \log x.$$

Example 4: *Solve* $x^3 \dfrac{d^3 y}{dx^3} + 2x^2 \dfrac{d^2 y}{dx^2} + 3x \dfrac{dy}{dx} - 3y = x^2 + x.$

(Meerut 2001, 07; Gorakhpur 05, 08; Kanpur 15)

Solution: Putting $x = e^z$ and denoting d/dz by D, the given differential equation becomes

$$[D(D-1)(D-2) + 2D(D-1) + 3D - 3]\, y = e^{2z} + e^z$$

or $$[D(D-1)(D-2) + 2D(D-1) + 3(D-1)]\, y = e^{2z} + e^z$$

or $$(D-1)\{D(D-2) + 2D + 3\}\, y = e^{2z} + e^z$$

or $$(D-1)(D^2+3)\,y = e^{2z} + e^{z}.$$

Hence the auxiliary equation is

$$(m-1)(m^2+3)=0\text{ , giving } m = 1, 0 \pm \sqrt{3}i.$$

$$\therefore \quad \text{C.F.} = c_1 e^{z} + e^{0z}(c_2 \cos\sqrt{3}z + c_3 \sin\sqrt{3}z)$$

$$= c_1 e^{z} + c_2 \cos\sqrt{3}z + c_3 \sin\sqrt{3}z.$$

Also $$\text{P.I.} = \frac{1}{(D-1)(D^2+3)}e^{2z} + \frac{1}{(D-1)(D^2+3)}e^{z}$$

$$= \frac{e^{2z}}{(2-1)(D^2+3)} + \frac{1}{(1^2+3)(D-1)}e^{z}\cdot 1$$

$$= \frac{1}{7}e^{2z} + \frac{1}{4}e^{z}\frac{1}{D+1-1}1$$

$$= \frac{1}{7}e^{2z} + \frac{1}{4}e^{z}\frac{1}{D}1$$

$$= \frac{1}{7}e^{2z} + \frac{1}{4}e^{z}.z\ .$$

$\therefore$ the complete solution is

$$y = c_1\, e^{z} + c_2 \cos\sqrt{3}z + c_3 \sin\sqrt{3}z + \frac{1}{7}e^{2z} + \frac{1}{4}ze^{z}$$

or $$y = c_1\, x + c_2 \cos(\sqrt{3}\log x) + c_3 \sin(\sqrt{3}\log x) + \frac{1}{7}x^2 + \frac{1}{4}x\log x\ .$$

$[\because\ e^{z} = x \text{ and } z = \log x]$

Example 5: *Solve* $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2 \log x$.

(Meerut 2004B, 10; Gorakhpur 07; Kanpur 07, 14; Lucknow 09, 11; Avadh 08; Bundelkhand 04; Garhwal 07, 10; Kumaun 09)

Solution: Putting $x = e^{z}$ or $z = \log x$ and denoting d/dz by D the equation becomes

$$[D(D-1) - D - 3]\,y = e^{2z}.z$$

or $$(D^2 - 2D - 3)\,y = e^{2z}.z\ .$$

$\therefore$ auxiliary equation is

$$(m^2 - 2m - 3) = 0$$

or $$(m-3)(m+1) = 0 \quad i.e.,\ m = -1, 3\ .$$

$$\therefore \quad \text{C.F.} = c_1 e^{3z} + c_2 e^{-z}.$$

And $$\text{P.I.} = \frac{1}{(D^2-2D-3)}e^{2z}.z$$

$$= e^{2z}.\frac{1}{[(D+2)^2 - 2(D+2) - 3]}z$$

$$= e^{2z} \frac{1}{(D^2 + 2D - 3)} z$$

$$= e^{2z} \frac{1}{-3[1 - \frac{2}{3} D - \frac{1}{3} D^2]} z$$

$$= e^{2z} - \frac{1}{3}\left[1 - \left(\frac{2}{3} D + \frac{1}{3} D^2\right)\right]^{-1} z$$

$$= -\frac{1}{3} e^{2z} \left(1 + \frac{2}{3} D + \frac{1}{3} D^2 + \dots\right) z$$

$$= -\frac{1}{3} e^{2z} (z + \frac{2}{3}).$$

$\therefore$ complete solution is

$$y = c_1 e^{3z} + c_2 e^{-z} - \frac{1}{3} e^{2z} (z + \frac{2}{3})$$

or $$y = c_1 x^3 + c_2 x^{-1} - \frac{1}{3} x^2 (\log x + \frac{2}{3}).$$

Comprehensive Exercise 1

Solve the following differential equations :

1. $x^2 (d^2 y/dx^2) - 4x (dy/dx) + 6y = x$. **(Garhwal 2005; Bundelkhand 10)**
2. $x^2 (d^2 y/dx^2) - 4x (dy/dx) + 6y = x^4$. **(Garhwal 2006; Gorakhpur 09; Kumaun 11)**
3. $x^2 (d^2 y/dx^2) - 2x (dy/dx) + 2y = 1/x$.
4. $x^2 (d^2 y/dx^2) + x (dy/dx) - 4y = x^2$. **(Rohilkhand 2009)**
5. $x^2 (d^2 y/dx^2) + 2x (dy/dx) - 20y = (x + 1)^2$.
6. $x^2 (d^2 y/dx^2) + 4x (dy/dx) + 2y = e^x$.

   (Meerut 2005; Avadh 08, 11; Rohilkhand 07, 11)
7. $x^2 (d^2 y/dx^2) + 7x (dy/dx) + 13y = \log x$. **(Bundelkhand 2006)**
8. **(i)** $x^2 (d^2 y/dx^2) - x (dy/dx) + 2y = x \log x$.

   (Rohilkhand 2010; Gorakhpur 06, 10, 11; Purvanchal 09; Avadh 06; Bundelkhand 13; Garhwal 13)

   (ii) $x^3 (d^3 y/dx^3) + 3x^2 (d^2 y/dx^2) + x (dy / dx) + y = x \log x$.

   (Kanpur 2009; Rohilkhand 08; Purvanchal 08)
9. $x^3 (d^3 y/dx^3) + 3x^2 (d^2 y/dx^2) + x (dy/dx) + y = x + \log x$. **(Avadh 2006, 14)**
10. $x^2 (d^2 y/dx^2) - x (dy/dx) + 4y = \cos (\log x) + x \sin (\log x)$. **(Garhwal 2009)**

11. $x^3 (d^3 y/dx^3) + 2x^2 (d^2 y/dx^2) + 2y = 10 [x + (1/x)]$.

(Meerut 2007B; Bundelkhand 09; Agra 2006; Lucknow 06; Garhwal 2010B)

12. $x^2 \dfrac{d^2 y}{dx^2} - 3x \dfrac{dy}{dx} + y = \dfrac{\log x \sin (\log x) + 1}{x}$.

13. $(x^2 D^2 + 3xD + 1)\ y = \dfrac{1}{(1-x)^2}$.

(Rohilkhand 2008; Purvanchal 08; Bundelkhand 01)

14. $2x^2 y \dfrac{d^2 y}{dx^2} + 4y^2 = x^2 \left(\dfrac{dy}{dx}\right)^2 + 2xy \left(\dfrac{dy}{dx}\right)$.

Answers 1

1. $y = c_1 x^2 + c_2 x^3 + \dfrac{1}{2} x$

2. $y = c_1 x^2 + c_2 x^3 + \dfrac{1}{2} x^4$

3. $y = c_1\ x + c_2\ x^2 + \dfrac{1}{6} \cdot (1/x)$

4. $y = c_1 x^2 + (c_2 / x^2) + \dfrac{1}{4} x^2 \log x$

5. $y = c_1 x^4 + c_2 x^{-5} - \dfrac{1}{14} x^2 - \dfrac{1}{9} x - \dfrac{1}{20}$

6. $y = (c_1 / x) + (c_2 / x^2) + (e^x / x^2)$

7. $y = c_1 \{\cos (\log x^2 + c_2)\} / x^3 + \dfrac{1}{13} \log x - \dfrac{6}{169}$

8. **(i)** $y = c_1\ x \cos (\log x + c_2) + x \log x$

(ii) $y = c_1 x^{-1} + c_2 \sqrt{x} \cos \{(\sqrt{3}/2) \log x + c_3\} + \dfrac{1}{2} x \log x - \dfrac{3}{4} x$

9. $y = c_1 x^{-1} + c_2 \sqrt{x} \cos \{(\sqrt{3}/2) \log x + c_3\} + \dfrac{1}{2} x + \log x$

10. $y = x\,[c_1 \cos (\sqrt{3} \log x) + c_2 \sin (\sqrt{3} \log x)]$

$$+ \frac{1}{13} [3 \cos (\log x) - 2 \sin (\log x)] + \frac{1}{2} x \sin (\log x)$$

11. $y = c_1 x^{-1} + c_2\ x \cos (\log x + c_3) + 5x + 2x^{-1} \log x$

12. $y = x^2 (c_1\ x^{\sqrt{3}} + c_2\ x^{-\sqrt{3}}) + \dfrac{1}{6x}$

$$+ \frac{1}{x} \left[\log x \cdot \frac{(5 \sin \log x + 6 \cos \log x)}{61} + \frac{382 \cos \log x + 54 \sin \log x}{3721} \right]$$

13. $y = (c_1 + c_2 \log x) \dfrac{1}{x} + \dfrac{1}{x} \log \dfrac{x}{1 - x}$

14. $y = x^2 (c_1 + c_2 \log x)^2$

3 Equations Reducible to Homogeneous Form

A differential equation of the form

$$(a + bx)^n \frac{d^n y}{dx^n} + P_1 (a + bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_{n-1} (a + bx) \frac{dy}{dx} + P_n y = Q,$$

where $P_1, P_2, \ldots, P_n$ are constants and Q is a function of x, can be reduced to the homogeneous linear form by putting $a + bx = t$ and then can be solved by the method explained in article 2. We can also solve this differential equation directly by making the substitution.

$$e^z = a + bx$$

or $$z = \log (a + bx).$$

If we denote the operator d/dz by D, we can easily see that

$$(a + bx) \frac{dy}{dx} = b\, Dy,$$

$$(a + bx)^2 \frac{d^2 y}{dx^2} = b^2 D (d - 1) y, \text{and so on.}$$

Illustrative Examples

Example 6: *Solve* $(x + a)^2 \dfrac{d^2 y}{dx^2} - 4 (x + a) \dfrac{dy}{dx} + 6 y = x$.

(Meerut 2001, 06, 11; Kanpur 11)

Solution: Putting $(x + a) = e^z$ and denoting d/dz by D, the given equation becomes

$$[D (D - 1) - 4D + 6] y = e^z - a$$

or $$(D - 3)(D - 2)\, y = e^z - a.$$

$\therefore$ auxiliary equation is $(m - 3)(m - 2) = 0$, giving $m = 3, 2$.

$\therefore$ $$\text{C.F.} = c_1 e^{3z} + c_2 e^{2z} = c_1 (e^z)^3 + c_2 (e^z)^2$$

$$= c_1 (x + a)^3 + c_2 (x + a)^2$$

and $$\text{P.I.} = \frac{1}{(D - 3)(D - 2)} e^z - \frac{1}{(D - 3)(D - 2)} a e^{0z}, \qquad [\because e^{0z} = 1]$$

$$= \frac{1}{(D - 3)(D - 2)} e^z - a \frac{1}{(D - 3)(D - 2)} e^{0z},$$

$$= \frac{1}{(1-3)(1-2)}\, e^{z} - a\,\frac{1}{(0-3)(0-2)}\, e^{0z}$$

$$= \frac{1}{2}\, e^{z} - \frac{1}{6}\, a$$

$$= \frac{1}{2}(x+a) - \frac{1}{6}\, a\,.$$

Hence the complete solution is $y = \text{C.F.} + \text{P.I.}$

or $$y = c_1 (x+a)^3 + c_2 (x+a)^2 + \frac{1}{2}(x+a) - \frac{1}{6}\, a\,.$$

Comprehensive Exercise 2

Solve the following differential equations :

1. $(3x+2)^2 \dfrac{d^2 y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1.$

(Bundelkhand 2009; Agra 06; Lucknow 06)

2. $(1+x)^2 \dfrac{d^2 y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos\log(1+x).$

3. $(1+2x)^2 \dfrac{d^2 y}{dx^2} = 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2.$

4. $(3x+2)^2 \dfrac{d^2 y}{dx^2} - (3x+2)\dfrac{dy}{dx} - 12y = 6x.$

5. $(x+1)^2 \dfrac{d^2 y}{dx^2} + (x+1)\dfrac{dy}{dx} = (2x+3)(2x+4).$

Answers 2

1. $y = c_1 (3x+2)^2 + c_2 (3x+2)^{-2} + \dfrac{1}{108}[(3x+2)^2 \log(3x+2) + 1]$

2. $y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) + 2\log(1+x)\sin\log(1+x)$

3. $y = (1+2x)^2 \log(1+2x)[1 + \log(1+2x)]$

4. $y = c_1 (3x+2)^2 + c_2 (3x+2)^{-2/3} - \dfrac{2}{15}(3x+2) + \dfrac{1}{3}$

5. $y = c_1 + c_2 \log(x+1) + (x+1)^2 + 6(x+1) + [\log(x+1)]^2$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The complementary function (C.F.) of the differential equation

$$x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - 3y = x^2 \log x \text{ is}$$

(a) $c_1 \cos x + c_2 \sin x$ (b) $c_1 e^{3z} + c_2 e^{-z}$

(c) $c_1 e^{z} + c_2 e^{-3z}$ (d) none of these

2. C.F. of the differential equation $x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x$ is

(a) $c_1 + c_2 x$ (b) $c_1 x^{-1} + c_2 x^{-2}$

(c) $c_1 e^{-x} + c_2 e^{x}$ (d) none of these

3. C.F. of the differential equation $x^4 \frac{d^3 y}{dx^3} + 2x^3 \frac{d^2 y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$ is

(a) $(c_1 + c_2 x) e^x + c_3 e^{-x}$ (b) $c_1 e^x + c_2 e^{-x} + c_3 e^{2x}$

(c) $(c_1 + c_2 \log x) x + \frac{c_3}{x} + \frac{1}{4x} \log x$ (d) none of these

4. C.F. of the differential equation $(x+a)^2 \frac{d^2 y}{dx^2} - 4(x+a) \frac{dy}{dx} + 6y = x$ is

(a) $c_1 x^2 + c_2 x^3$ (b) $c_1 (x+a)^3 + c_2 (x+a)^2$

(c) $c_1 e^{3x} + c_2 e^{2x}$ (d) none of these

5. To solve homogeneous linear differential equation, we put :

(a) $x = X + h, y = Y + k$ (b) $y = \frac{1}{x}$

(c) $z = \log x$ (d) none of these **(Garhwal 2008)**

6. The substitution $z = \log x$ transforms the differential equation into the following :

$$x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 4y = 0$$

(a) $\frac{d^2 y}{dz^2} - 4 \frac{dy}{dz} + 4y = 0$ (b) $4 \frac{d^2 y}{dz^2} - \frac{dy}{dz} + 4y = 0$

(c) $\frac{d^2 y}{dz^2} - 3 \frac{dy}{dz} + 4y = 0$ (d) $z^2 \frac{d^2 y}{dz^2} - 3z \frac{dy}{dz} + 4y = 0$

(Garhwal 2009)

7. The general solution of the differential equation $x^2 \frac{d^2 y}{dx^2} - x\frac{dy}{dx} + y = 0$

(a) $Ax + Bx^2$ (b) $Ax + B \log x$

(c) $Ax + Bx^2 \log x$ (d) $Ax + Bx \log x$

where A and B are constants. **(Kumaun 2015)**

8. On putting $x = e^z$, the transformal differential equation of $x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = x$ is

(a) $\frac{d^2 y}{dz^2} + y = e^z$ (b) $\frac{d^2 y}{dx^2} - y = e^z$

(c) $\frac{d^2 y}{dx^2} + y = e^x$ (d) $\frac{d^2 y}{dx^2} - y = e^{-z}$ **(Kumaun 2016)**

9. The substitutions in solving the homogeneous linear equation are

(a) $z = e^x, x\frac{dy}{dx} = Dy, x^2 \frac{d^2 y}{dx^2} = D(D-1)y$, where $D \equiv \frac{d}{dz}$

(b) $x = e^{-z}, x\frac{dy}{dx} = Dy, x^2 \frac{d^2 y}{dx^2} = D(D-1)y$, where $D \equiv \frac{d}{dz}$

(c) $x = e^z, x\frac{dy}{dx} = Dy, x^2 \frac{d^2 y}{dx^2} = D(D-1)y$, where $D \equiv \frac{d}{dz}$

(d) none of these

10. The auxiliary roots of the equation $(x^3 D^3 + 2x^2 D^2 + 3xD - 3)\, y = x^2 + x$ are

(a) $m = -1, \pm \sqrt{3}\, i$ (b) $m = \pm 1, \sqrt{3}\, i$

(c) $m = 1, \pm \sqrt{3}\, i$ (d) $m = \pm 1, -\sqrt{3}\, i$

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. Homogeneous linear differential equation $x^2 \frac{d^2 y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0$

will reduce to a linear differential equation with constant coefficients by putting $x =$ ……

2. To reduce the homogeneous linear differential equation

$$(x+a)^3 \frac{d^3 y}{dx^3} - 2(x+a)^2 \frac{d^2 y}{dx^2} - 5(x+a)\frac{dy}{dx} + 3y = 2x$$

to a linear differential equation with constant coefficients, we have to put ……

True or False

Write 'T' for true and 'F' for false statement.

1. $x^2 \dfrac{d^2 y}{dx^2} - x \dfrac{dy}{dx} - 3y = x^2 \log x$ is a linear homogeneous differential equation of order two. **(Meerut 2003)**
2. $x^4 \dfrac{d^3 y}{dx^3} + 2x^3 \dfrac{d^2 y}{dx^2} - x^2 \dfrac{dy}{dx} + xy = 1$ is not a linear homogeneous differential equation of order 3.
3. $x^3 \dfrac{d^3 y}{dx^3} + 4x^2 \dfrac{d^2 y}{dx^2} + 2x \dfrac{dy}{dx} + xy = x^2$ is a linear homogeneous differential equation of order 3.

Answers

Multiple Choice Questions

1.	(b)	**2.**	(b)	**3.**	(c)	**4.**	(b)	**5.**	(c)
6.	(a)	**7.**	(d)	**8.**	(a)	**9.**	(c)	**10.**	(c)

Fill in the Blank(s)

1. e^z **2.** $(x + a) = e^z$

True or False

1. *T* **2.** *F* **3.** *F*

☐

Chapter

6

Ordinary Simultaneous Differential Equations

1 Introduction

In the present chapter, we shall discuss differential equations containing **one independent variable and two or more than two dependent variables**. To completely solve such equations we need as many simultaneous equations as there are dependent variables.

2 Methods of Solving Simultaneous Linear Differential Equations with Constant Coefficients

Let x and y be the two dependent variables and t be the independent variable. Thus the equations will contain differential coefficients of x, y with respect to t. Let $D \equiv d / dt$. Then such equations can be put in the form

$$f_1(D)\, x + f_2(D)\, y = T_1 \qquad \ldots(1)$$

and

$$\phi_1(D)\, x + \phi_2(D)\, y = T_2, \qquad \ldots(2)$$

where T_1 and T_2 are some functions of the independent variable t. Here $f_1(D), f_2(D), \phi_1(D)$ and $\phi_2(D)$ are all rational integral functions of D with constant coefficients.

There are two methods to solve such equations.

First method: Method of Elimination (Symbolic method):

To eliminate y between (1) and (2), operating both sides of (1) by $\phi_2(D)$ and of (2) by by $f_2(D)$, we get

$$\phi_2(D)\, f_1\,(D)\, x+\phi_2\,(D)\, f_2\,(D)\, y=\phi_2\,(D)\, T_1 \qquad \ldots(3)$$

and
$$f_2\,(D)\,\phi_1\,(D)\, x+f_2\,(D)\,\phi_2\,(D)\, y=f_2\,(D)\, T_2. \qquad \ldots(4)$$

Subtracting (4) from (3), we get

$$[\phi_2\,(D)\, f_1\,(D)-f_2\,(D)\,\phi_1\,(D)]\, x=\phi_2\,(D)\, T_1-f_2\,(D)\, T_2,$$

which is of the form $F\,(D)\,x=T$. ...(5)

The equation (5) is a linear differential equation with constant coefficients in x and t. Solving it we can find the value of x in terms of t.

Putting this value of x in either (1) or (2), we get the value of y.

Note 1: The equations can also be solved by first finding the value of y by eliminating x between (1) and (2) and then solving the resulting linear differential equation in y and t. The value of x can now be obtained from (1) or (2) after putting the value of y.

Note 2: Since $f_2(D)$ and $\phi_2(D)$ are functions of D with constant coefficients, so

$$f_2(D)\,\phi_2\,(D)=\phi_2\,(D)\, f_2\,(D).$$

Second method: Method of differentiation:

Sometimes x or y can be conveniently eliminated if we differentiate the equations (1) and (2). For example suppose the two given equations (1) and (2) connect four quantities $x, y, dx/dt$ and dy/dt. Differentiating (1) and (2) w.r.t. t, we get in all four equations containing $x, y, \frac{dx}{dt}, \frac{dy}{dt}, \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$. Eliminating three quantities $y, \frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ from these four equations, we get an equation of the second order with x as the dependent and t as the independent variable. Solving this equation we find the value of x in terms of t. Then the value of the other variable can be found.

3 Number of Arbitrary Constants

In the general solution of (1) and (2) the number of arbitrary constants is equal to the degree of D in the determinant

$$\Delta=\begin{vmatrix} f_1(D) & f_2(D) \\ \phi_1(D) & \phi_2(D) \end{vmatrix}, \text{ if } \Delta\neq 0.$$

If Δ vanishes then the system is dependent. Here we shall not consider such cases.

Illustrative Examples

Example 1: *Solve the simultaneous equations*

$$\frac{dx}{dt} - 7x + y = 0\ ,\ \frac{dy}{dt} - 2x - 5y = 0\ .$$

(Purvanchal 2008; Rohilkhand 09; Lucknow 10; Kumaun 11; Kanpur 15)

Solution: Writing D for d / dt, the given equations are

$$(D - 7)\, x + y = 0 \qquad \ldots(1)$$

and $$-2x + (D - 5)\, y = 0\ . \qquad \ldots(2)$$

Let us eliminate y between (1) and (2). Operating on both sides of (1) by $D - 5$ and then subtracting (2) from it, we get

$$\{(D - 5)(D - 7) + 2\}\, x = 0$$

or $$(D^2 - 12D + 37)\, x = 0, \qquad \ldots(3)$$

which is a linear differential equation of second order in x and t with constant coefficients.

A.E. is $m^2 - 12m + 37 = 0 \Rightarrow m = 6 \pm i$.

$\therefore$ the solution of (3) is $x = e^{6t}\,(c_1 \cos t + c_2 \sin t)$ ...(4)

$$\therefore \quad \frac{dx}{dt} = 6\, e^{6t}\,(c_1 \cos t + c_2 \sin t) + e^{6t}\,(-c_1 \sin t + c_2 \cos t)\ .$$

Putting the values of x and dx / dt in (1), we get

$$\begin{aligned} y &= 7x - Dx = 7x - (dx / dt) \\ &= 7\, e^{6t}\,(c_1 \cos t + c_2 \sin t) - 6\, e^{6t}\,(c_1 \cos t + c_2 \sin t) \\ &\qquad - e^{6t}\,(-c_1 \sin t + c_2 \cos t) \end{aligned}$$

or $$y = e^{6t}\,[(c_1 - c_2) \cos t + (c_1 + c_2) \sin t]\ . \qquad \ldots(5)$$

The required general solution of the given equations consists of the equations (4) and (5).

Example 2: *Solve the simultaneous equations*

$$\frac{d^2 y}{dt^2} - 3x - 4y = 0\ ,\ \frac{d^2 y}{dt^2} + x + y = 0\ .$$

(Kanpur 2003; Meerut 09)

Solution: Writing D for (d/dt), the given equations are

$$(D^2 - 3)\, x - 4y = 0 \qquad \ldots(1)$$

and $$x + (D^2 + 1)\, y = 0\ . \qquad \ldots(2)$$

Let us eliminate y from (1) and (2). Multiplying both sides of (2) by 4 and operating on both sides of (1) by $D^2 + 1$ and adding, we get

$$\{(D^2 + 1)(D^2 - 3) + 4\}\, x = 0$$

or $$(D^4 - 2D^2 + 1)\, x = 0. \qquad \ldots(3)$$

A.E. is $\quad m^4 - 2m^2 + 1 = 0 \quad$ or $\quad (m^2 - 1)^2 = 0.$

$\therefore \quad m = \pm 1, \pm 1.$

$\therefore$ the solution of (3) is $x = (c_1 + c_2\, t)\, e^t + (c_3 + c_4 t)\, e^{-t}$ ...(4)

$$\therefore \quad \frac{dx}{dt} = (c_1 + c_2 t)\, e^t - (c_3 + c_4 t)\, e^{-t} + c_2 e^t + c_4 e^{-t},$$

$$\text{and} \quad \frac{d^2 x}{dt^2} = (c_1 + c_2 t)\, e^t + (c_3 + c_4 t)\, e^{-t} + 2c_2\, e^t - 2c_4\, e^{-t}.$$

Putting these values in (1), we get

$$\begin{aligned} 4y &= D^2 x - 3x = (d^2 x/dt^2) - 3x \\ &= (c_1 + c_2 t)\, e^t + (c_3 + c_4 t)\, e^{-t} \\ &\qquad + 2\, c_2 e^t - 2c_4 e^{-t} - 3\, (c_1 + c_2 t)\, e^t - 3\, (c_3 + c_4 t)\, e^{-t} \\ &= -\, 2\, (c_1 + c_2 t)\, e^t - 2\, (c_3 + c_4 t)\, e^{-t} + 2c_2\, e^t - 2c_4 e^{-t} \end{aligned}$$

$$\text{or} \quad y = \frac{1}{2}\, e^t\, (c_2 - c_1 - c_2 t) - \frac{1}{2}\, e^{-1}\, (c_4 + c_3 + c_4 t). \qquad \text{...(5)}$$

The required general solution of the given equations consists of the equations (4) and (5).

Example 3: *Solve the simultaneous equations*

$$\frac{d^2 x}{dt^2} + 4x + y = te^{3t}, \frac{d^2 y}{dt^2} + y - 2x = cos^2 t.$$

Solution: Writing D for (d/dt), the given equations are

$$(D^2 + 4)\, x + y = te^{3t} \qquad \text{...(1)}$$

$$\text{and} \quad -\, 2x + (D^2 + 1)\, y = \cos^2 t \qquad \text{...(2)}$$

Eliminating y from (1) and (2), we get

$$\{(D^2 + 1)\, (D^2 + 4) + 2\}\, x = (D^2 + 1)\, (te^{3t}) - \cos^2 t$$

$$\text{or} \quad (D^4 + 5D^2 + 6)\, x = 10te^{3t} + 6e^{3t} - \cos^2 t. \qquad \text{...(3)}$$

A.E. is $\quad m^4 + 5m^2 + 6 = 0 \quad$ or $\quad (m^2 + 3)\, (m^2 + 2) = 0\,.$

$\therefore \quad m = \pm\, i\sqrt{3},\ \pm\, i\sqrt{2}\,.$

$\therefore \quad$ C.F. $= (c_1 \cos \sqrt{3}t + c_2 \sin \sqrt{3}t) + (c_3 \cos \sqrt{2}t + c_4 \sin \sqrt{2}t)\,.$

$$\begin{aligned} \text{P. I.} &= \frac{10}{D^4 + 5D^2 + 6}\, te^{3t} + \frac{6}{D^4 + 5D^2 + 6}\, e^{3t} - \frac{1}{D^4 + 5D^2 + 6} \cos^2 t \\ &= 10e^{3t}\, \frac{1}{(D + 3)^4 + 5\, (D + 3)^2 + 6}\, t + 6\, \frac{1}{3^4 + 5\,.3^2 + 6} \cdot e^{3t} \\ &\qquad - \frac{1}{D^4 + 5D^2 + 6}\, \frac{1}{2}\, (1 + \cos 2t) \\ &= 10e^{3t}\, \frac{1}{132 + 138D + 52D^2 + \ldots}\, t + \frac{1}{22}\, e^{3t} \\ &\qquad - \frac{1}{6 + 5D^2 + D^4}\, \frac{1}{2} - \frac{1}{D^4 + 5D^2 + 6} \left(\frac{1}{2} \cos 2t\right) \end{aligned}$$

$$=10e^{3t}\cdot\frac{1}{132}\left(1+\frac{23}{22}D+\dots\right)^{-1}t+\frac{1}{22}e^{3t}$$

$$-\frac{1}{2}\cdot\frac{1}{6+5D^2+D^4}e^{0t}-\frac{1}{2}\frac{\cos 2t}{(-4)^2+5(-4)+6}$$

$$=\frac{5e^{3t}}{66}\left(t-\frac{23}{22}\right)+\frac{e^{3t}}{22}-\frac{1}{12}-\frac{1}{4}\cos 2t\,.$$

∴ $$x=(c_1\cos\sqrt{3}t+c_2\sin\sqrt{3}t)+(c_3\cos\sqrt{2}t+c_4\sin\sqrt{2}t)$$
$$+\frac{5}{66}te^{3t}-\frac{49}{1452}e^{3t}-\frac{1}{4}\cos 2t-\frac{1}{12}\cdot \qquad \dots(4)$$

Now from (4), we find $\dfrac{dx}{dt}$ and $\dfrac{d^2x}{dt^2}$.

Putting the values of x and d^2x/dt^2 in (1), we get

$$y=-\frac{d^2x}{dt^2}-4x+te^{3t}$$

or $$y=-(c_1\cos\sqrt{3}t+c_2\sin\sqrt{3}t)-2(c_3\cos\sqrt{2}t+c_4\sin\sqrt{2}t)$$
$$+\frac{1}{66}t\,.\,e^{3t}-\frac{33}{2452}e^{3t}+\frac{1}{3}\cdot \qquad \dots(5)$$

The required solution consists of the equations (4) and (5).

Example 4: *Solve the simultaneous differential equations*

$$x\frac{dy}{dx}+z=0 \qquad \dots(1) \qquad\qquad x\frac{dz}{dx}+y=0\,. \qquad \dots(2)$$

(Meerut 2001, 09B; Kumaun13)

Solution: Differentiating (1) with respect to x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{dz}{dx}=0 \quad \text{or} \quad x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+x\frac{dz}{dx}=0\,.$$

Putting the value of $x\dfrac{dz}{dx}$ from (2), we get

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0\,, \qquad \dots(3)$$

which is a linear homogeneous equation.

To solve (3), putting $x=e^t$ and denoting d/dt by D, we have

$$x\frac{dy}{dx}=Dy,\ x^2\frac{d^2y}{dx^2}=D(D-1)\,y\,.$$

∴ the equation (3) becomes

$$\{D(D-1)+D-1\}\,y=0 \quad \text{or} \quad (D^2-1)\,y=0\,.$$

∴ $$y=c_1e^t+c_2e^{-t}=c_1x+c_2x^{-1}. \qquad \dots(4)$$

∴ $$\frac{dy}{dx}=c_1-\frac{c_2}{x^2}\Rightarrow x\frac{dy}{dx}=c_1\,x-\frac{c_2}{x}\cdot$$

$\therefore$ from (1), we get $z = -x\dfrac{dy}{dx} = -c_1x + c_2x^{-1}$. ...(5)

The required solution consists of the equations (4) and (5).

Example 5: *Solve the simultaneous differential equations*

$$\frac{d^2x}{dt^2} + m^2y = 0,\ \frac{d^2y}{dt^2} - m^2x = 0.$$

Solution: Writing D for (d/dt), the given equations are

$$D^2x + m^2y = 0 \quad \text{...(1)}$$

and $$D^2y - m^2x = 0. \quad \text{...(2)}$$

Eliminating y from (1) and (2), we get

$$(D^4 + m^4)x = 0.$$

A.E. is $M^4 + m^4 = 0 \Rightarrow (M^2 + m^2)^2 - 2M^2m^2 = 0$.

or $$(M^2 - \sqrt{2}Mm + m^2)(M^2 + \sqrt{2}Mm + m^2) = 0.$$

$\therefore$ $$M^2 - \sqrt{2}Mm + m^2 = 0, \quad \text{or} \quad M^2 + \sqrt{2}Mm + m^2 = 0.$$

Solving these, we get

$$M = \frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}} \quad \text{and} \quad M = -\frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}.$$

$\therefore$ $$x = c_1e^{(m/\sqrt{2})t}\cos\left(\frac{m}{\sqrt{2}}t + c_2\right) + c_3e^{-(m/\sqrt{2})t}\cos\left(\frac{m}{\sqrt{2}}t + c_4\right) \quad \text{...(3)}$$

$\therefore$ $$\frac{dx}{dt} = c_1e^{(m/\sqrt{2})t}.\frac{m}{\sqrt{2}}\left\{\cos\left(\frac{m}{\sqrt{2}}t + c_2\right) - \sin\left(\frac{m}{\sqrt{2}}t + c_2\right)\right\}$$
$$-c_3e^{-(m/\sqrt{2})t}.\frac{m}{\sqrt{2}}\left\{\cos\left(\frac{m}{\sqrt{2}}t + c_4\right) + \sin\left(\frac{m}{\sqrt{2}}t + c_4\right)\right\}$$

and $$\frac{d^2x}{dt^2} = c_1e^{(m/\sqrt{2})t}.\frac{m^2}{2}\left\{\cos\left(\frac{m}{\sqrt{2}}t + c_2\right) - 2\sin\left(\frac{m}{\sqrt{2}}t + c_2\right)\right.$$
$$\left. - \cos\left(\frac{m}{\sqrt{2}}t + c_2\right)\right\} + c_3e^{-(m/\sqrt{2})t}.\frac{m^2}{2}\left\{\cos\left(\frac{m}{\sqrt{2}}t + c_4\right)\right.$$
$$\left. + 2\sin\left(\frac{m}{\sqrt{2}}t + c_4\right) - \cos\left(\frac{m}{\sqrt{2}}t + c_4\right)\right\}$$
$$= m^2\left\{c_3e^{-(m/\sqrt{2})t}\sin\left(\frac{m}{\sqrt{2}}t + c_4\right) - c_1e^{(m/\sqrt{2})t}\sin\left(\frac{m}{\sqrt{2}}t + c_2\right)\right\}.$$

From the equation (1), we get $y = -\dfrac{1}{m^2}\dfrac{d^2x}{dt^2}$.

Putting the value of d^2x/dt^2, we have

$$y = c_1e^{(m/\sqrt{2})t}\sin\left(\frac{m}{\sqrt{2}}t + c_2\right) - c_3e^{-(m/\sqrt{2})t}.\sin\left(\frac{m}{\sqrt{2}}t + c_4\right). \quad \text{...(4)}$$

The equations (3) and (4) constitute the required general solution of the given equations.

Example 6: *Solve* $t\,dx = (t - 2x)\,dt,\ t\,dy = (tx + ty + 2x - t)\,dt$. **(Rohilkhand 2001)**

Solution: The given equations are

$$t\,dx = (t - 2x)\,dt \qquad \ldots(1)$$

and

$$t\,dy = (tx + ty + 2x - t)\,dt. \qquad \ldots(2)$$

From (1), we have $\dfrac{dx}{dt} + \dfrac{2}{t}x = 1$, which is linear.

$$\text{I.F.} = e^{\int (2/t)dt} = e^{2\log t} = t^2.$$

$$\therefore \quad xt^2 = \int t^2 \cdot 1\,dt + c_1 = \frac{1}{3}t^3 + c_1$$

or

$$x = \frac{1}{3}t + c_1 t^{-2}. \qquad \ldots(3)$$

Now adding (1) and (2), we get

$$t\,(dx + dy) = t\,(x + y)\,dt \quad \text{or} \quad \frac{dx + dy}{x + y} = dt.$$

Integrating, $\log(x + y) = t + \log c_2$.

$$\therefore \quad x + y = c_2 e^t \ \text{ or } \ y = c_2 e^t - x. \qquad \ldots(4)$$

Putting in (4) the value of x found in (3), we get

$$y = c_2 e^t - \frac{1}{3}t - c_1 t^{-2}. \qquad \ldots(5)$$

The equations (3) and (5) give the required general solution of the given equations.

Example 7: *Solve the simultaneous equations*

$$t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} + 2y = 0,\ t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - 2x = 0.$$

Solution: Here both the given equations are linear homogeneous equations. First we shall change them into linear differential equations with constant coefficients.

For this, we put $t = e^z$.

Then denoting d/dz by D, we have

$$t\frac{dx}{dt} = Dx,\ t^2\frac{d^2x}{dt^2} = D(D-1)\,x,\ t\frac{dy}{dt} = Dy,\ t^2\frac{d^2y}{dt^2} = D(D-1)\,y.$$

$\therefore$ the given equations transform to

$$\{D(D-1) + D\}\,x + 2y = 0$$

i.e.,

$$D^2x + 2y = 0 \qquad \ldots(1)$$

and

$$\{D(D-1) + D\}\,y - 2x = 0$$

i.e.,

$$D^2y - 2x = 0. \qquad \ldots(2)$$

Eliminating y from (1) and (2), we get

$$(D^4 + 4)\,x = 0.$$

A.E. is $m^4 + 4 = 0$, or $(m^2 + 2)^2 - 4m^2 = 0$

or

$$(m^2 - 2m + 2)(m^2 + 2m + 2) = 0.$$

$\therefore$ $$m^2 - 2m + 2 = 0 \quad \text{or} \quad m^2 + 2m + 2 = 0 .$$

$\therefore$ $$m = \frac{2 \pm \sqrt{(4-8)}}{2} \quad \text{and} \quad m = \frac{-2 \pm \sqrt{(4-8)}}{2}$$

i.e., $$m = 1 \pm i, -1 \pm i .$$

$\therefore$ $$x = e^z (c_1 \cos z + c_2 \sin z) + e^{-z} (c_3 \cos z + c_4 \sin z) \qquad \ldots(3)$$

$\therefore$ $$\frac{dx}{dz} = e^z (c_1 \cos z + c_2 \sin z) + e^z (-c_1 \sin z + c_2 \cos z)$$
$$- e^{-z} (c_3 \cos z + c_4 \sin z) + e^{-z} (-c_3 \sin z + c_4 \cos z)$$

and $$\frac{d^2 x}{dz^2} = e^z (c_1 \cos z + c_2 \sin z) + 2e^z (-c_1 \sin z + c_2 \cos z)$$
$$- e^z (c_1 \cos z + c_2 \sin z) + e^{-z} (c_3 \cos z + c_4 \sin z)$$
$$- 2e^{-z} (-c_3 \sin z + c_4 \cos z) - e^{-z} (c_3 \cos z + c_4 \sin z)$$
$$= 2e^z (-c_1 \sin z + c_2 \cos z) - 2e^{-z} (-c_3 \sin z + c_4 \cos z) .$$

From (1), we have

$$y = -\frac{1}{2} D^2 x = -\frac{1}{2} \frac{d^2 x}{dz^2} \cdot$$

$\therefore$ $$y = e^z (c_1 \sin z - c_2 \cos z) + e^{-z} (-c_3 \sin z + c_4 \cos z) . \qquad \ldots(4)$$

Now changing the variable z into t in the equations (3) and (4) by the relation $t = e^z$ or $z = \log t$, we get

$$x = t (c_1 \cos \log t + c_2 \sin \log t) + t^{-1} (c_3 \cos \log t + c_4 \sin \log t)$$

and $$y = t (c_1 \sin \log t - c_2 \cos \log t) + t^{-1} (-c_3 \sin \log t + c_4 \cos \log t) .$$

These equations give the required general solution of the given equations.

Example 8: *Solve* $\dfrac{dx}{dt} = ny - mz$, $\dfrac{dy}{dt} = lz - nx$, $\dfrac{dz}{dt} = mx - ly$. **(Kanpur 2012)**

Solution: Multiplying the given equations by x, y, z respectively and then adding, we get

$$x \frac{dx}{dt} + y \frac{dy}{dt} + z \frac{dz}{dt} = 0 .$$

Integrating, we get

$$\frac{1}{2} x^2 + \frac{1}{2} y^2 + \frac{1}{2} z^2 = k$$

or $$x^2 + y^2 + z^2 = c_1 . \qquad \ldots(1)$$

Now multiplying the given equations by l, m, n respectively and then adding, we get

$$l \frac{dx}{dt} + m \frac{dy}{dt} + n \frac{dz}{dt} = 0 .$$

Integrating, we get $lx + my + nz = c_2$ . ...(2)

The equations (1) and (2) constitute the complete solution of the given differential equations.

Example 9: *Solve the simultaneous equations*

$$3\frac{dx}{dt}+2\frac{dy}{dt}-4x+3y=8e^{-3t}, 4\frac{dx}{dt}+\frac{dy}{dt}+3x+4y=8e^{-3t}$$

given that $x=\frac{1}{5}$, $y=0$, *when* $t=0$.

Solution: Writing D for (d/dt), the given equations can be written as

$$(3D-4)\,x+(2D+3)\,y=8\,e^{-3t} \qquad \ldots(1)$$

and $$(4D+3)\,x+(D+4)\,y=8\,e^{-3t} \qquad \ldots(2)$$

Eliminating y from (1) and (2), we get

$$[(D+4)(3D-4)-(2D+3)(4D+3)]\,x$$
$$=8\,[(D+4)\,e^{-3t}-(2D+3)\,e^{-3t}]$$

or $$-5\,(D^2+2D+5)\,x=8\,(-3e^{-3t}+4e^{-3t}+6e^{-3t}-3e^{-3t})$$

or $$(D^2+2D+5)\,x=-\frac{32}{5}\,e^{-3t}.$$

A.E. is $m^2+2m+5=0$, giving $m=-1\pm 2i$.

$\therefore$ $$\text{C.F.}=e^{-1}\,(c_1\cos 2t+c_2\sin 2t).$$

Also $$\text{P.I.}=\frac{1}{D^2+2D+5}\left(-\frac{32}{5}\,e^{-3t}\right)=-\frac{4}{5}\,e^{-3t}.$$

$\therefore$ $$x=e^{-t}\,(c_1\cos 2t+c_2\sin 2t)-\frac{4}{5}\,e^{-3t}.$$

Given that $x=1/5$, when $t=0$, we have

$$\frac{1}{5}=c_1-\frac{4}{5} \quad \text{or} \quad c_1=1.$$

$\therefore$ $$x=e^{-t}\,(\cos 2t+c_2\sin 2t)-\frac{4}{5}\,e^{-3t}$$

$\Rightarrow$ $$\frac{dx}{dt}=-e^{-t}\,(\cos 2t+c_2\sin 2t)+e^{-t}\,(-2\sin 2t+2c_2\cos 2t)+\frac{12}{5}\,e^{-3t}.$$

Now multiplying (2) by 2 and then subtracting (1) from it, we get

$$5y=-5\frac{dx}{dt}-10x+8e^{-3t}.$$

Putting the values of x and dx/dt, we get

$$y=e^{-t}(-\cos 2t-2c_2\cos 2t+2\sin 2t-c_2\sin 2t)+\frac{4}{5}\,e^{-3t}.$$

Given that $y=0$, when $t=0$, we have

$$0=-1-2c_2+\frac{4}{5}, \quad \text{or} \quad c_2=-\frac{1}{10}.$$

$\therefore$ $$x=\frac{1}{10}\,e^{-t}\,(10\cos 2t-\sin 2t)-\frac{4}{5}\,e^{-3t} \qquad \ldots(3)$$

and $$y=\frac{1}{10}\,e^{-t}\,(21\sin 2t-8\sin 2t)+\frac{4}{5}\,e^{-3t}. \qquad \ldots(4)$$

The required solution consists of equations (3) and (4).

Comprehensive Exercise 1

Solve the following simultaneous differential equations :

1. $\dfrac{dx}{dt} + 5x + y = e^t,\ \dfrac{dy}{dt} - x + 3y = e^{2t}.$ **(Meerut 2004B)**

2. $\dfrac{dx}{dt} + 2\dfrac{dy}{dt} - 2x + 2y = 3e^t,\ 3\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = 4e^{2t}.$ **(Meerut 2007)**

3. $\dfrac{dx}{dt} + 2x - 3y = t,\ \dfrac{dy}{dt} - 3x + 2y = e^{2t}.$ **(Lucknow 2011)**

4. $\left(\dfrac{d}{dt} + 2\right) x + 3y = 0\ ,\ 3x + \left(\dfrac{d}{dt} + 2\right) y = 2e^{3t}.$

5. $\dfrac{dx}{dt} = 3x + 2y\ ,\ \dfrac{dy}{dt} + 5x + 3y = 0\ .$ **(Gorakhpur 2009; Kumaun 12)**

6. $\dfrac{dx}{dt} = ax + by\ ,\ \dfrac{dy}{dt} = a'x + b'y.$

7. $\dfrac{dx}{dt} + \omega y = 0\ ,\ \dfrac{dy}{dt} - \omega x = 0\ .$ **(Kanpur 2003, 05; Kumaun 06)**

8. $\dfrac{dx}{dt} + 4x + 3y = t,\ \dfrac{dy}{dt} + 2x + 5y = e^t.$

9. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2y = 2\cos t - 7\sin t,\ \dfrac{dx}{dt} - \dfrac{dy}{dt} + 2x = 4\cos t - 3\sin t\ .$ **(Lucknow 2005, 09; Avadh 10)**

10. $(D - 17)\ y + (2D - 8)\ z = 0,\ (13D - 53)\ y - 2z = 0\ .$

11. $4\dfrac{dx}{dt} + 9\dfrac{dy}{dt} + 11x + 31y = e^t,\ 3\dfrac{dx}{dt} + 7\dfrac{dy}{dt} + 8x + 24y = e^{2t}.$

12. $\dfrac{dx}{dt} + 2\dfrac{dy}{dt} + x + 7y = e^t - 3\ ,\ \dfrac{dy}{dt} - 2x + 3y = 12 - 3e^t.$

13. $\dfrac{d^2x}{dt^2} + 4x + y = te^t,\ \dfrac{d^2y}{dt^2} + y - 2x = \sin^2 t.$

14. $\dfrac{d^2x}{dt^2} - \dfrac{dy}{dt} = 2x + 2t\ ,\ \dfrac{dx}{dt} + 4\dfrac{dy}{dt} = 3y\ .$

15. $\dfrac{dx}{dt} + \dfrac{2}{t}(x - y) = 1,\ \dfrac{dy}{dt} + \dfrac{1}{t}(x + 5y) = t\ .$

Answers 1

1. $x = (c_1 + c_2 t)\, e^{-4t} + \dfrac{4}{25} e^t - \dfrac{1}{36} e^{2t},\ y = -\,(c_1 + c_2 + c_2 t)\, e^{-4t} + \dfrac{7}{36} e^{2t} + \dfrac{1}{25} e^t$

2. $x = \dfrac{1}{2} e^{2t} - \dfrac{3}{11} e^t + c_1\, e^{-6t/5},\ y = \dfrac{15}{22} e^t - 8c_1\, e^{-6t/5} + c_2 e^{-t}$

3. $x = c_1\, e^{-5t} + c_2 e^t + \dfrac{3}{7} e^{2t} - \dfrac{2}{5} t - \dfrac{13}{25},\ y = -\,c_1 e^{-5t} + c_2 e^t + \dfrac{4}{7} e^{2t} - \dfrac{3}{5} t - \dfrac{12}{25}$

4. $x = c_1\, e^t + c_2 e^{-5t} - \dfrac{3}{8} e^{3t},\ y = -\, c_1 e^t + c_2 e^{-5t} + \dfrac{5}{8} e^{3t}$

5. $x = c_1 \cos t + c_2 \sin t,\ y = \dfrac{1}{2}(c_2 - 3c_1)\cos t - \dfrac{1}{2}(c_1 + 3c_2)\sin t$

6. $x = c_1 e^{m_1 t} + c_2 e^{m_2 t},\ y = \dfrac{1}{b}\left\{(m_1 - a)\, c_1\, e^{m_1 t} + (m_2 - a)\, c_2 e^{m_2 t}\right\}$, where

$$m_1 = \frac{(a + b') + \sqrt{[(a - b')^2 + 4a' b]}}{2} \text{ and } m_2 = \frac{(a + b') + \sqrt{[(a - b')^2 + 4a' b]}}{2}$$

7. $x = c_1 \cos \omega t + c_2 \sin \omega t,\ y = c_1 \sin \omega t - c_2 \cos \omega t$

8. $x = c_1 e^{-2t} + c_2 e^{-7t} + \dfrac{5}{14} t - \dfrac{1}{8} e^t - \dfrac{31}{196}$,

$$y = \frac{1}{3}\left\{-\frac{3}{7} t - 2c_1\, e^{-2t} + 3c_2\, e^{-7t} + \frac{5}{8} e^t + \frac{27}{98}\right\}$$

9. $x = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + 3\cos t$,

$$y = (\sqrt{2} + 1)\, c_1 e^{\sqrt{2}t} + (1 - \sqrt{2})\, c_2\, e^{-\sqrt{2}t} + 2\sin t$$

10. $y = c_1 e^{3x} + c_2 e^{5x},\ z = -\,7c_1 e^{3x} + 6c_2 e^{5x}$

11. $y = (c_1 + c_2 t)\, e^{-4t} + \dfrac{31}{25} e^t - \dfrac{49}{36} e^{2t},\ y = -\,(c_2 + c_1 + c_2 t)\ e^{-4t} + \dfrac{19}{36} e^{2t} - \dfrac{11}{25} e^t$.

12. $x = \dfrac{1}{2} e^{-4t}\,[(c_2 - c_1)\cos t - (c_1 + c_2)\sin t] + \dfrac{31}{26} e^t - \dfrac{93}{17}$.

$$y = e^{-4t}\,(c_1 \cos t + c_2 \sin t) + \frac{6}{17} - \frac{2}{13} e^t$$

13. $x = c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t + c_3 \cos \sqrt{3}t + c_4 \sin \sqrt{3}t + \dfrac{e^t}{6}\left(t - \dfrac{1}{6}\right) - \dfrac{1}{12} + \dfrac{\cos 2t}{4}$.

$$y = -\,2\,(c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t) - (c_3 \cos \sqrt{3}t + c_4 \sin \sqrt{3}t) + \frac{1}{6} e^t \left(t - \frac{7}{6}\right) + \frac{1}{3}.$$

14. $x = (c_1 + c_2 t)\, e^t + c_3\, e^{-3/2t} - t,\ y = (3c_2 - c_1 - c_2 t)\, e^t - \dfrac{1}{6} c_3\, e^{-(3/2)t} - \dfrac{1}{3}$.

15. $x = \dfrac{c_1}{t^3} + \dfrac{c_2}{t^4} + \dfrac{t^2}{15} + \dfrac{3t}{10},\ y = -\dfrac{c_1}{2t^3} - \dfrac{c_2}{t^4} + \dfrac{2t^2}{15} - \dfrac{1}{20} t$.

4 Simultaneous Equations of the Form

$$P_1\, dx + Q_1\, dy + R_1\, dz = 0 \qquad \ldots(1)$$

$$P_2\, dx + Q_2\, dy + R_2\, dz = 0, \qquad \ldots(2)$$

where the coefficients P_1, P_2, Q_1, Q_2, R_1 *and* R_2 *are functions of* x, y, z .

Above is the general form of a system of simultaneous equations of the first order in three variables.

Taking z as the independent variable, the equations (1) and (2) can be written in the form

$$P_1 \frac{dx}{dz} + Q_1 \frac{dy}{dz} + R_1 = 0$$

$$P_2 \frac{dx}{dz} + Q_2 \frac{dy}{dz} + R_2 = 0 .$$

Solving these equations for dx/dz and dy/dz by the method of cross- multiplication, we get

$$\frac{dx/dz}{Q_1 R_2 - Q_2 R_1} = \frac{dy/dz}{R_1 P_2 - R_2 P_1} = \frac{1}{P_1 Q_2 - P_2 Q_1}$$

or $$\frac{dx}{Q_1 R_2 - Q_2 P_1} = \frac{dy}{R_1 P_2 - R_2 P_1} = \frac{dz}{P_1 Q_2 - P_2 Q_1},$$

which is of the form

$$\boldsymbol{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}, \qquad \ldots(3)$$

where P, Q, R are functions of x, y and z.

Thus the equations (1) and (2) can always be expressed in the form (3). To solve the simultaneous equations in the form (3) we can use the following methods.

First Method: Suppose one variable is absent from two members of (3), say, z is absent from P and Q. Then the solution of

$$\frac{dx}{P} = \frac{dy}{Q}$$

gives a relation between x and y and we thus get one equation constituting the complete solution of the given equations. If necessary, we can use this relation to eliminate x or y from one of the remaining equations in (3) and we can find the other integral relation which gives us the second equation of the complete solution.

Second Method: By a property of ratio and proportion in algebra, we know that each of the equal fractions $\frac{dx}{P}, \frac{dy}{Q}, \frac{dz}{R}$ in (3), is also equal to

$$\frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}, \frac{L\,dx + M\,dy + N\,dz}{LP + MQ + NR}, \text{ etc.}$$

$$\therefore \quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = \frac{L\,dx + M\,dy + N\,dz}{LP + MQ + NR}. \qquad \ldots(4)$$

The relations between x, y, z that satisfy (3) also satisfy (4) .

By a proper choice of multipliers $l, m, n; L, M, N$ etc., it is possible to find equations which can be easily solved. Particularly l, m, n can be so chosen that

$$lP + mQ + nR = 0.$$

which makes $l\,dx + m\,dy + n\,dz = 0$. $\qquad \ldots(5)$

Let $l\,dx + m\,dy + n\,dz$ be an exact differential, say du.

Then (5) $\Rightarrow du = 0 \Rightarrow u = a$, which gives one equation constituting the complete solution.

Again L, M, N may be found such that

$$LP + MQ + NR = 0 \quad \Rightarrow \quad L\,dx + M\,dy + N\,dz = 0 .$$

If $L\,dx + M\,dy + N\,dz$ be an exact differential, say dv, then $dv = 0 \Rightarrow v = b$, which gives the second equation constituting the complete solution.

Hence the complete solution consists of the equations $u = a$ and $v = b$ which must of course be independent of each other.

Note 1: Sometimes only one set of multipliers may serve the purpose.

Note 2: We can find one relation, say $u = a$, by the first method and the second relation, say $v = b$, by the second method.

5 Geometrical Interpretation of the Differential Equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$.

(Gorakhpur 2005)

From solid geometry, we know that the direction cosines of the tangent to a curve in three dimensional Euclidean space at any point (x, y, z) are

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$$

i.e., the direction cosines are in the ratio of $dx : dy : dz$. Hence the given differential equations represent a system of curves in space, the direction cosines of the tangent to any member of this system at any point (x, y, z) are proportional to P, Q and R. Let $u = a$ and $v = b$ be two component solutions of the given equations. Then the curves are obtained by the intersection of the surfaces $u = a$ and $v = b$. Since the arbitrary constants a and b can take infinite values, the curves are doubly infinite in number.

Illustrative Examples

Example 10: *Solve* $\dfrac{x\,dx}{z^2 - 2yz - y^2} = \dfrac{dy}{y + z} = \dfrac{dz}{y - z}$.

(Meerut 2001, 04; Avadh 08; Garhwal 08; Kumaun 13)

Solution: Taking the last two members of the given equations, we get

$$(y - z)\,dy = (y + z)\,dz \quad \text{or} \quad y\,dy - z\,dz - (y\,dz + z\,dy) = 0$$

or
$$2y\,dy - 2z\,dz - 2\,(y\,dz + z\,dy) = 0 .$$

Integrating, we get

$$y^2 - z^2 - 2yz = c_1 . \qquad \ldots(1)$$

Again choosing 1, y, z as multipliers, we get

$$\frac{x\,dx}{z^2 - 2yz - y^2} = \frac{dy}{y+z} = \frac{dz}{y-z}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{1.(z^2 - 2yz - y^2) + y\,(y+z) + z\,(y-z)} = \frac{x\,dx + y\,dy + z\,dz}{0}.$$

$\therefore$ $x\,dx + y\,dy + z\,dz = 0$.

Integrating, we get $x^2 + y^2 + z^2 = c_2$ . ...(2)

The required solution consists of the equations (1) and (2).

Example 11: *Solve* $\dfrac{dx}{y^2 + z^2 - x^2} = \dfrac{dy}{-2xy} = \dfrac{dz}{-2xz}$. **(Garhwal 2002; Kumaun 10)**

Solution: Taking the last two members of the given equations, we get

$$\frac{dy}{y} = \frac{dz}{z}.$$

Integrating, we get

$$\log\,y = \log\,z + \log\,c_1$$

or $\log\,y = \log\,(c_1\,z)$ or $y = c_1 z$ . ...(1)

Again choosing x, y, z as multipliers, we get

$$\frac{dx}{y^2 + z^2 - x^2} = \frac{dy}{-2\,xy} = \frac{dz}{-2xz} = \frac{x\,dx + y\,dy + z\,dz}{-x\,(x^2 + y^2 + z^2)}.$$

Taking the last two fractions, we get

$$\frac{dz}{z} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2}.$$

Integrating, $\log\,z + \log\,c_2 = \log\,(x^2 + y^2 + z^2)$

or $x^2 + y^2 + z^2 = zc_2$ . ...(2)

The required solution consists of the equations (1) and (2).

Example 12: *Solve* $\dfrac{dx}{1} = \dfrac{dy}{-2} = \dfrac{dz}{3x^2\,sin\,(y+2x)}$.

(Meerut 2001, 10B; Gorakhpur 07; Garhwal 06; Kumaun 06)

Solution: From the first two members, we get

$$2x + y = c_1.$$

Taking the first and the last members, we get

$$\frac{dx}{1} = \frac{dz}{3x^2\,\sin\,(y+2x)} = \frac{dz}{dx^2\,\sin\,c_1}, \text{using (1)}.$$

$\therefore$ $dz = 3x^2\,\sin c_1\,dx$.

Integrating, $z = x^3\,\sin c_1 + c_2$ or $z - x^3\,\sin\,(y+2x) = c_2$ . ...(2)

The required solution consists of the equations (1) and (2).

Example 13: *Solve* $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{-y(z^2 + x^2)} = \dfrac{dz}{z(x^2 + y^2)}$.

(Meerut 2005, 11)

Solution: Choosing $\dfrac{1}{x}, -\dfrac{1}{y}, -\dfrac{1}{z}$ as multipliers, we have

$$\text{each fraction} = \frac{\dfrac{dx}{x} - \dfrac{dy}{y} - \dfrac{dz}{z}}{0}.$$

$$\therefore \qquad \frac{dx}{x} - \frac{dy}{y} - \frac{dz}{z} = 0 \qquad \Rightarrow \qquad \frac{dx}{x} = \frac{dy}{y} + \frac{dz}{z}.$$

Integrating, $\log x + \log c_1 = \log y + \log z$

or $\qquad yz = c_1 x.$ ...(1)

Again choosing x, y, z as multipliers, we get

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dz}{0}.$$

$$\therefore \qquad x\,dx + y\,dy + z\,dz = 0.$$

Integrating, $\qquad x^2 + y^2 + z^2 = c_2.$ ...(2)

The equations (1) and (2) give the required solution of the given equations.

Example 14: *Solve* $\dfrac{dx}{\cos(x+y)} = \dfrac{dy}{\sin(x+y)} = \dfrac{dz}{z}$.

(Meerut 2006, 07B; Gorakhpur 08, 11; Garhwal 11)

Solution: From the given equations, we have

$$\frac{dx + dy}{\cos(x+y) + \sin(x+y)} = \frac{dx - dy}{\cos(x+y) - \sin(x+y)} = \frac{dz}{z}.$$

Taking the first two members, we get

$$\frac{\cos(x+y) - \sin(x+y)}{\cos(x+y) + \sin(x+y)}(dx + dy) = dx - dy.$$

Integrating, $\log[\cos(x+y) + \sin(x+y)] = x - y + \log c_1$

or $\qquad [\cos(x+y) + \sin(x+y)]\,e^{y-x} = c_1.$ ...(1)

Again taking the first and the third members, we get

$$\frac{dx + dy}{\cos(x+y) + \sin(x+y)} = \frac{dz}{z} \qquad \text{or} \qquad \frac{dz}{z} = \frac{(1/\sqrt{2})(dx + dy)}{\sin(x + y + \frac{1}{4}\pi)}$$

or $\qquad \sqrt{2}\,\dfrac{dz}{z} = \operatorname{cosec}(x + y + \frac{1}{4}\pi)(dx + dy).$

Integrating, $\qquad \sqrt{2}\log z = \log\tan\frac{1}{2}(x + y + \frac{1}{4}\pi) + \log c_2$

or $\qquad z^{\sqrt{2}}\cot\left(\dfrac{x}{2} + \dfrac{y}{2} + \dfrac{\pi}{8}\right) = c_2.$ ...(2)

The required solution consists of the equations (1) and (2).

Example 15: Solve $\dfrac{dx}{x}=\dfrac{dy}{y}=\dfrac{dz}{z-a\sqrt{(x^2+y^2+z^2)}}$.

(Meerut 2005B, 10B)

Solution: From the first two members of the given differential equations, we have

$$x=c_1 y. \qquad \ldots(1)$$

Again choosing x, y, z as multipliers, we have

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z-a\sqrt{(x^2+y^2+z^2)}}=\frac{x\,dx+y\,dy+z\,dz}{x^2+y^2+z^2-az\sqrt{(x^2+y^2+z^2)}}$$

or $$\frac{dy}{y}=\frac{dz}{z-au}=\frac{u\,du}{u^2-azu}, \text{ putting } x^2+y^2+z^2=u^2$$

or $$\frac{dy}{y}=\frac{dz}{z-au}=\frac{du}{u-az}=\frac{du+dz}{(u+z)(1-a)}.$$

Taking the first and the last members and integrating, we get

$$(1-a)\log y=\log(u+z)+\log c_2 \qquad \text{or} \qquad y^{(1-a)}=(u+z)c_2$$

or $$y^{(1-a)}=c_2\{\sqrt{(x^2+y^2+z^2)}+z\}. \qquad \ldots(2)$$

The required solution consists of the equations (1) and (2).

Example 16. Solve $\dfrac{dx}{y^3x-2x^4}=\dfrac{dy}{2y^4-x^3y}=\dfrac{dz}{9z(x^3-y^3)}$.

Solution: From the first two members of the given differential equations, we get

$$(2y^4-x^3y)\,dx=(y^3x-2x^4)\,dy$$

or $$\left(\frac{2y}{x^3}-\frac{1}{y^2}\right)dx=\left(\frac{1}{x^2}-\frac{2x}{y^3}\right)dy, \text{ dividing by } x^3y^3$$

or $$\left(\frac{1}{x^2}\,dy-\frac{2y}{x^3}\,dx\right)+\left(\frac{1}{y^2}\,dx-\frac{2x}{y^3}\,dy\right)=0.$$

Integrating, $$\frac{y}{x^2}+\frac{x}{y^2}=c_1. \qquad \ldots(1)$$

Again taking $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{3z}$ as multipliers, we get

$$\text{each fraction}=\frac{\frac{1}{x}\,dx+\frac{1}{y}\,dy+\frac{1}{3z}\,dz}{(y^3-2x^3)+(2y^3-x^3)+3(x^3-y^3)}=\frac{\frac{1}{x}\,dx+\frac{1}{y}\,dy+\frac{1}{3z}\,dz}{0}.$$

$$\therefore \quad \frac{1}{x}\,dx+\frac{1}{y}\,dy+\frac{1}{3z}\,dz=0.$$

Integrating, $$\log x+\log y+\frac{1}{3}\log z=\log c_2$$

or $$xyz^{1/3}=c_2. \qquad \ldots(2)$$

The required solution consists of the equations (1) and (2).

Example 17: *Solve* $\dfrac{dx}{xy} = \dfrac{dy}{y^2} = \dfrac{dz}{zxy - 2x^2}$.

(Purvanchal 2006; Garhwal 09)

Solution: From the first two members of the given differential equations, we get

$$\frac{dx}{x} = \frac{dy}{y}.$$

Integrating, $\log x = \log y + \log c_1$

or $x = c_1 y.$ ...(1)

Again taking the last two members, we get

$$\frac{dy}{y^2} = \frac{dz}{xyz - 2x^2} \quad \text{or} \quad \frac{dy}{y^2} = \frac{dz}{c_1 y^2 z - 2c_1^2 y^2} \qquad [\because x = c_1 y]$$

or $dy = \dfrac{dz}{c_1 z - 2c_1^2}$ or $c_1\, dy = \dfrac{dz}{z - 2c_1}$.

Integrating, $c_1 y = \log (z - 2c_1) + c_2$

or $x = \log \left(z - 2\dfrac{x}{y} \right) + c_2$ $[\because x = c_1 y]$

or $x = \log (yz - 2x) - \log y + c_2.$ ...(2)

The required solution consists of the equations (1) and (2).

Example 18: *Solve* $\dfrac{dx}{x^2 + y^2} = \dfrac{dy}{2xy} = \dfrac{dz}{(x + y) z}$.

(Kumaun 2009)

Solution: From the given equations, we have

$$\frac{dx + dy}{x^2 + y^2 + 2xy} = \frac{dx - dy}{x^2 + y^2 - 2xy} = \frac{dz}{(x + y) z}.$$

Taking the first two members, we have

$$\frac{dx + dy}{(x + y)^2} = \frac{dx - dy}{(x - y)^2}.$$

Integrating, $-(x + y)^{-1} = -(x - y)^{-1} + c_1$

or $\dfrac{1}{x - y} - \dfrac{1}{x + y} = c_1,$ or $\dfrac{2y}{x^2 - y^2} = c_1$

or $2y = c_1 (x^2 - y^2)$ ...(1)

Taking the first and the last members, we have

$$\frac{dx + dy}{(x + y)^2} = \frac{dz}{(x + y) z} \quad \text{or} \quad \frac{dx + dy}{x + y} = \frac{dz}{z}.$$

Integrating, $\log (x + y) = \log z + \log c_2$

or $(x + y) = c_2 z.$ ...(2)

The required solution consists of the equations (1) and (2).

Example 19: *Solve* $\dfrac{dx}{x^2}=\dfrac{dy}{y^2}=\dfrac{dz}{nxy}\cdot$

Solution: We have $\dfrac{dx}{x^2}=\dfrac{dy}{y^2}$.

Integrating, $\dfrac{1}{x}=\dfrac{1}{y}+c_1$ or $y-x=c_1\,xy$. ...(1)

Also using $1/x, -1/y$ and c_1/n as multipliers, we have

$$\text{each fraction} = \frac{(1/x)\,dx-(1/y)\,dy+(c_1/n)\,dz}{x-y+c_1xy}$$

$$= \frac{(1/x)\,dx-(1/y)\,dy+(c_1/n)\,dz}{0}, \text{ using (1).}$$

$\therefore$ $(1/x)\,dx-(1/y)\,dy+(c_1/n)\,dz=0$.

Integrating, $\log x-\log y+(c_1/n)\,z=c_2$

or $\dfrac{c_1}{n}z=\log\left(\dfrac{y}{x}\right)+c_2$ or $z=\dfrac{nxy}{y-x}\log\left(\dfrac{y}{x}\right)+c_2$. ...(2)

The required solution consists of the equations (1) and (2).

Comprehensive Exercise 2

Solve the following simultaneous differential equations :

1. $\dfrac{dx}{yz}=\dfrac{dy}{zx}=\dfrac{dz}{xy}\cdot$ **(Kumaun 2008, 12)**
2. $\dfrac{dx}{mz-ny}=\dfrac{dy}{nx-lz}=\dfrac{dz}{ly-mx}$.

   Also give geometrical interpretation of the solution of these equations.
3. $\dfrac{dx}{z}=\dfrac{dy}{-z}=\dfrac{dz}{z^2+(x+y)^2}\cdot$
4. $\dfrac{a\,dx}{(b-c)\,yz}=\dfrac{b\,dy}{(c-a)\,zx}=\dfrac{c\,dz}{(a-b)xy}\cdot$ **(Lucknow 2006; Kumaun 11)**
5. $\dfrac{dx}{xz\,(z^2+xy)}=\dfrac{dy}{-yz\,(z^2+xy)}=\dfrac{dz}{x^4}\cdot$
6. $\dfrac{dx}{z\,(x+y)}=\dfrac{dy}{z\,(x-y)}=\dfrac{dz}{x^2+y^2}\cdot$
7. $\dfrac{-dx}{x\,(x+y)}=\dfrac{dy}{y\,(x+y)}=\dfrac{dz}{(x-y)\,(2x+2y+z)}\cdot$

8. $\dfrac{dx}{1}=\dfrac{dy}{3}=\dfrac{dz}{5z+\tan(y-3x)}\cdot$ **(Meerut 2006B; Gorakhpur 08, 11)**

9. $\dfrac{dx}{x(y-z)}=\dfrac{dy}{y(z-x)}=\dfrac{dz}{z(x-y)}\cdot$

10. $\dfrac{dx}{x(y^2-z^2)}=\dfrac{dy}{y(z^2-x^2)}=\dfrac{dz}{z(x^2-y^2)}\cdot$ **(Gorakhpur 2005, 09)**

11. $\dfrac{dx}{1+y}=\dfrac{dy}{1+x}=\dfrac{dz}{z}\cdot$

12. $\dfrac{dx}{y^2+yz+z^2}=\dfrac{dy}{z^2+zx+x^2}=\dfrac{dz}{x^2+xy+y^2}\cdot$ **(Avadh 2007)**

$$\left[\textbf{Hint. }\frac{dy-dx}{(x-y)(x+y+z)}=\frac{dz-dy}{(y-z)(x+y+z)}=\frac{dz-dx}{(x-z)(x+y+z)}\right]$$

13. $\dfrac{dx}{x^2-yz}=\dfrac{dy}{y^2-zx}=\dfrac{dz}{z^2-xy}\cdot$ **(Gorakhpur 2006; Garhwal 01, 10B; Kumaun 15)**

$$\left[\textbf{Hint. }\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}=\frac{dz-dx}{(z-x)(x+y+z)}\right]\cdot$$

14. $\dfrac{dx}{x^2+y^2+yz}=\dfrac{dy}{x^2+y^2-xz}=\dfrac{dz}{z(x+y)}\cdot$ **(Kanpur 2009)**

$$\left[\textbf{Hint. }\frac{dx-dy}{z(x+y)}=\frac{dz}{z(x+y)}=\frac{x\,dx+y\,dy}{(x+y)(x^2+y^2)}\right]$$

15. $\dfrac{dx}{y+z}=\dfrac{dy}{z+x}=\dfrac{dz}{x+y}\cdot$

$$\left[\textbf{Hint. }\frac{dx-dy}{y-x}=\frac{dy-dz}{z-x}=\frac{dx+dy+dz}{2(x+y+z)}\right]\cdot$$

Answers 2

1. $x^2-y^2=c_1,\ x^2-z^2=c_2$
2. $lx+my+nz=c_1,\ x^2+y^2+z^2=c_2$

   These equations constituting the general solution of the given differential equations represent a family of circles.
3. $x+y=c_1,\ \log\{z^2+(x+y)^2\}-2x=c_2$
4. $a^2x^2+b^2y^2+c^2z^2=c_1,\ ax^2+by^2+cz^2=c_2$
5. $xy=c_1,\ (z^2+xy)^2-x^4=c_2$

6. $x^2 - y^2 - z^2 = c_1,\ 2xy - z^2 = c_2$
7. $xy = c_1,\ x^2 + y^2 + (x + y)\, z = c_2$
8. $y - 3x = c_1, 5z + \tan(y - 3x) = c_2\, e^{5x}$
9. $x + y + z = c_1,\ xyz = c_2$
10. $x^2 + y^2 + z^2 = c_1,\ xyz = c_2$
11. $z\,(x - y) = c_1,\ z = c_2\,(x + y + 2)$
12. $(y - x) = c_1\,(z - y), (y - x) = c_2\,(z - x)$
13. $(x - y) = c_1\,(y - z),\ (y - z) = c_2\,(z - x)$
14. $x - y - z = c_1,\ x^2 + y^2 = z^2 c_2$
15. $(y - x) = c_1\,(z - y),\ (x - y)^2\,(x + y + z) = c_2$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Solving the equations $\dfrac{dx}{dt} + 5x + y = e^t,\ \dfrac{dy}{dt} - x + 3y = e^{2t}$, for x, we get $x =$

   (a) $(c_1 + c_2 t)\, e^{-4t} + \dfrac{4}{25} e^t - \dfrac{1}{36} e^{2t}$ (b) $c_1\, e^{-4t} + c_2\, e^t - \dfrac{2}{5}$

   (c) $c_1 e^t + c_2 e^{-4t} + c_3 e^{-2t}$ (d) none of these.

2. Solving the equations $\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = 0,\ \dfrac{dy}{dt} + 5x + 3y = 0$, we get $x =$

   (a) $c_1\, e^t + c_2 e^{-t}$ (b) $c_1 \cos t + c_2 \sin t$

   (c) $(c_1 + c_2 t) \cos t$ (d) $c_1\, e^t + c_2 e^{-t}$. **(Garhwal 2008)**

3. Solving the equations $\dfrac{dx}{dt} + \omega y = 0,\ \dfrac{dy}{dt} - \omega x = 0$ for x, we get $x =$

   (a) $c_1 e^{\omega t} + c_2 e^{-\omega t}$ (b) $(c_1 + c_2 t)\, e^{\omega t}$

   (c) $c_1 \cos \omega t + c_2 \sin \omega t$ (d) none of these. **(Kumaun 2008)**

4. Solution set of the equations $\dfrac{dx}{y + z} = \dfrac{dy}{z + x} = \dfrac{dz}{x + y}$ is

   (a) $(y - x) = c_1\,(z - y)$
   $(x - y)^2\,(x + y + z) = c_2$

   (b) $x + y = c_1\,(y + z)$
   $(x - y)^2\,(x + y + z) = c_2$.

   (c) $(y - z) = c_1\,(x - y)$
   $(x + y)^2\,(x - y + z) = c_2$

   (d) none of these.

5. Solution set of the equations $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$ is

(a) $xy=c_1, yz=c_2$ (b) $x=c_1 z, y=c_2 z$

(c) $x=c_1 y, y=c_2 z$ (d) none of these.

(Kumaun 2006; Garhwal 11)

6. Solution set of the equations $\frac{dx}{yz}=\frac{dy}{zx}=\frac{dz}{xy}$ is

(a) $x^2+y^2=c_1, y^2-z^2=c_2$ (b) $x^2-z^2=c_1, y^2+z^2=c_2$

(c) $x^2-y^2=c_1, x^2-z^2=c_2$ (d) none of these. **(Kumaun 2007)**

7. Solution set of the simultaneous equations $\frac{dx}{y}=\frac{dy}{x}=\frac{dz}{z}$ is

(a) $x^2-y^2=c_1, x+y=c_2 z$ (b) $x^2+y^2=c_1, x+y=c_2 z$

(c) $x^2-y^2=c_1, x-y=c_2 z$ (d) $x^2-y^2=c_1, y^2-z^2=c_2$

8. In equation $\frac{dx}{dt}-7x+y=0, \frac{dy}{dt}-2x-5y=0$, value of x is

(a) $e^{6t}(c_1\cos t+c_2\sin t)$ (b) $e^{-6t}(c_1\cos t+c_2\sin t)$

(c) $e^{t}(c_1\cos 6t+c_2\sin 6t)$ (d) none of these

(Garhwal 2007, 10B)

9. Solution of the differential equation $dx=dy=dz$ is

(a) $x-y=c_1$ and $x-z=c_2$ (b) $x+y=c_1$ and $x-z=c_2$

(c) $x-y=c_1$ and $x+z=c_2$ (d) $x+y=c_1$ and $x+z=c_2$

10. If $\frac{dx}{dt}=3x+2y, \frac{dy}{dt}=5x+3y$ then x is equal to

(a) $c_1\cosh\sqrt{10}\,t+c_2\sinh\sqrt{10}\,t$ (b) $e^{3t}(c_1\cosh\sqrt{10}\,t+c_2\sinh\sqrt{10}\,t)$

(c) $e^{-3t}(c_1\cosh\sqrt{10}\,t+c_2\sinh\sqrt{10}\,t)$ (d) $e^{3t}(c_1\cosh\sqrt{10}\,t-c_2\sinh\sqrt{10}\,t)$

(Kanpur 2016)

11. If $\frac{dx}{dt}+7x-y=0, \frac{dy}{dt}+2x+5y=0$ then y is equal to

(a) $e^{-6t}(c_1\cos t+c_2\sin t)$ (b) $e^{6t}(c_1\cos t+c_2\sin t)$

(c) $\frac{e^{-6t}}{2}(c_1\cos t+c_2\sin t)$ (d) $\frac{e^{6t}}{2}(c_1\cos t+c_2\sin t)$

12. Solution set of the equations $\frac{x\,dx}{y^2 z}=\frac{dy}{xz}=\frac{dz}{y^2}$ is

(a) $x^3+y^3=c_1, x^2-z^2=c_2$ (b) $x^3-y^3=c_1, x^2+z^2=c_2$

(c) $x^3-y^3=c_1, x^2-z^2=c_2$ (d) $x^3+y^3=c_1, x^2+z^2=c_2$

13. Solution set of the equations $\dfrac{dx}{1}=\dfrac{dy}{-2}=\dfrac{dz}{3x^2\sin(y+2x)}$ is

(a) $2x+y=c_1, z-x^3\sin(y+2x)=c_2$

(b) $2x-y=c_1, z-x^3\sin(y+2x)=c_2$

(c) $2x+y=c_1, z+x^3\sin(y+2x)=c_2$

(d) $2x-y=c_1, z+x^3\sin(y+2x)=c_2$

14. Solving the equations $\dfrac{dx}{dt}-y=t, \dfrac{dy}{dt}+x=1$ for x, we get $x=$

(a) $c_1 e^{-2t}+c_2 e^{2t}$ (b) $(c_1+c_2 x)e^t$

(c) $c_1\cos t+c_2\sin t+2$ (d) $c_1\cos 2t+c_2\sin 2t+2$

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. Eliminating y between the equations

$$\frac{dx}{dt}-7x+y=0, \frac{dy}{dt}-2x-5y=0$$

we obtain the differential equation…… .

2. Eliminating y between the simultaneous equations

$$\frac{d^2x}{dt^2}+m^2y=0, \frac{d^2y}{dt^2}-m^2x=0$$

we obtain the differential equation …… . **(Meerut 2003)**

3. The complete solution of the simultaneous equations

$$\frac{dx}{dt}=ny-mz, \frac{dy}{dt}=lz-nx, \frac{dz}{dt}=mx-ly \text{ is …… .}$$

Answers

Multiple Choice Questions

1. (a)	**2.** (b)	**3.** (c)	**4.** (a)	**5.** (b)
6. (c)	**7.** (a)	**8.** (a)	**9.** (a)	**10.** (b)
11. (a)	**12.** (c)	**13.** (a)	**14.** (c)	

Fill in the Blank(s)

1. $(D^2-12D+37)x=0$ **2.** $(D^4+m^4)x=0$

3. $x^2+y^2+z^2=c_1, lx+my+nz=c_2$

□

Chapter

7

Linear Equations of Second Order with Variable Coefficients

1 Linear Equation of Second Order

An equation of the form

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = R,$$

where P, Q, R are functions of x only, is called the '*linear equation of second order*'.

There is no general method for solving such equations. Here we shall discuss certain methods by which the solutions of such equations can be found.

2 The Complete Solution in Terms of a Known Integral

The following theorem relating to the linear differential equation of the second order is of great importance :

If an integral included in the complementary function of such an equation be known, then the complete primitive (or the general solution) can be found in terms of the known integral.

Let $y = u$ be a known integral in the complementary function of

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad \ldots(1)$$

i.e., it is a solution of

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = 0\,.$$

$$\therefore \qquad \frac{d^2 u}{dx^2} + P\frac{du}{dx} + Qu = 0\,. \qquad \ldots(2)$$

Let $y = uv$ be the solution of (1).

Putting $y = uv$, we get $\dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

and $$\frac{d^2 y}{dx^2} = v\frac{d^2 u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2 v}{dx^2}\cdot$$

Substituting these values the equation (1) becomes

$$\left(v\frac{d^2 u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2 v}{dx^2}\right) + P\left(v\frac{du}{dx} + u\frac{dv}{dx}\right) + Q\,vu = R$$

or $$u\frac{d^2 v}{dx^2} + \frac{dv}{dx}\left(2\frac{du}{dx} + Pu\right) + v\left(\frac{d^2 u}{dx^2} + P\frac{du}{dx} + Qu\right) = R$$

or $$u\frac{d^2 v}{dx^2} + \frac{dv}{dx}\left(2\frac{du}{dx} + Pu\right) + v.0 = R\text{, using (2)}$$

or $$\frac{d^2 v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}\cdot \qquad \ldots(3)$$

Putting $\dfrac{dv}{dx} = p,\ \dfrac{d^2 v}{dx^2} = \dfrac{dp}{dx}$, (3) becomes

$$\frac{dp}{dx} + \left(P + \frac{2}{u}\frac{du}{dx}\right)p = \frac{R}{u}, \qquad \ldots(4)$$

which is linear with p as dependent variable.

$$\text{I.F.} = e^{\int\left(P + \frac{2}{u}\frac{du}{dx}\right)dx} = e^{\int\left(P\,dx + \frac{2}{u}\,du\right)} = e^{2\log u + \int P\,dx} = u^2 e^{\int P\,dx}.$$

Hence solution of (4) is

$$pu^2 e^{\int P\,dx} = \int\left[\frac{R}{u}u^2\,e^{\int P\,dx}\right]dx + c_1. \qquad \ldots(5)$$

$$\therefore \qquad p = \frac{dv}{dx} = \frac{c_1 e^{-\int P\,dx}}{u^2} + \frac{e^{-\int P\,dx}}{u^2}\int u\,Re^{\int P\,dx}dx.$$

Integrating this, we get

$$y = c_2 + c_1\int\frac{e^{-\int P\,dx}}{u^2}dx + \int\left[\frac{e^{-\int P\,dx}}{u^2}\int u\,Re^{\int P\,dx}dx\right]dx.$$

Hence the solution of (1) is

$$y = uv = c_2 u + c_1\,u\int\frac{e^{-\int P\,dx}}{u^2}dx + u\int\left[\frac{e^{-\int P\,dx}}{u^2}\cdot\int u\,Re^{\int P\,dx}dx\right]dx. \qquad \ldots(6)$$

It contains the given solution $y = u$ and since it contains two arbitrary constants so it is the complete primitive or the general solution of (1). It is evident from (6) that the second part of the complementary function is

$$u\int \frac{e^{-\int P\,dx}}{u^2}\,dx \text{ and the particular integral is}$$

$$u\int \left[\frac{e^{-\int P\,dx}}{u^2}\cdot\int u\,Re^{\int P\,dx}dx\right]dx.$$

3 To Find one Integral In C.F. by Inspection

Consider $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0. \qquad \ldots(1)$$

1. $y = e^{mx}$ is a solution of (1) if

$$m^2 + Pm + Q = 0\,.$$

We have $y = e^{mx} \Rightarrow \dfrac{dy}{dx} = me^{mx}, \dfrac{d^2y}{dx^2} = m^2e^{mx}$.

If $y = e^{mx}$ is a solution of (1), then

$$(m^2 + Pm + Q)\,e^{mx} = 0 \Rightarrow m^2 + Pm + Q = 0\,.$$

Deductions. (i) $y = e^x$ is a solution of (1) if $1 + P + Q = 0$.

(ii) $y = e^{-x}$ is a solution of (1) if $1 - P + Q = 0$.

2. $y = x^m$ is a solution of (1) if $m\,(m-1) + Pmx + Qx^2 = 0$.

We have $y = x^m \Rightarrow \dfrac{dy}{dx} = mx^{m-1}, \dfrac{d^2y}{dx^2} = m\,(m-1)\,x^{m-2}$.

If $y = x^m$ is a solution of (1), then

$$m\,(m-1)\,x^{m-2} + Pmx^{m-1} + Qx^m = 0$$

or $$m\,(m-1) + Pmx + Qx^2 = 0\,.$$

Deductions. (i) $y = x$ is a solution of (1) if $P + Qx = 0$.

(ii) $y = x^2$ is a solution of (1) if $2 + 2Px + Qx^2 = 0$.

All the above results are summarised below :

(i) $y = e^x$ is a part of C.F. if $1 + P + Q = 0$.

(ii) $y = e^{-x}$ is a part of C.F. if $1 - P + Q = 0$.

(iii) $y = e^{mx}$ is a part of C.F. if $m^2 + m\,P + Q = 0$.

(iv) $y = x$ is a part of C.F. if $P + Qx = 0$.

(v) $y = x^2$ is a part of C.F. if $2 + 2Px + Qx^2 = 0$.

(vi) $y = x^m$ is a part of C.F. if $m\,(m-1) + Pmx + Qx^2 = 0$.

Illustrative Examples

Example 1: *Solve* $(x+2)\dfrac{d^2y}{dx^2}-(2x+5)\dfrac{dy}{dx}+2y=(x+1)e^x$.

(Gorakhpur 2009; Lucknow 07; Rohilkhand 09; Kanpur 07, 09; Garhwal 09; Kumaun 11)

Solution: The given equation can be written as

$$\frac{d^2y}{dx^2}-\frac{(2x+5)}{x+2}\frac{dy}{dx}+\frac{2}{x+2}y=\frac{(x+1)}{x+2}e^x. \qquad \ldots(1)$$

Comparing (1) with the standard form

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R,\ \text{we have}$$

$$P=-\frac{2x+5}{x+2},\ Q=\frac{2}{x+2},\ R=\frac{x+1}{x+2}e^x.$$

Here $2^2+2P+Q=0$,

$\therefore$ $y=e^{2x}$ is a part of the C.F. of the solution of the equation (1).

Putting $y=ve^{2x}$, $\dfrac{dy}{dx}=e^{2x}\dfrac{dv}{dx}+2v\,e^{2x}$, and $\dfrac{d^2y}{dx^2}=e^{2x}\dfrac{d^2v}{dx^2}+4e^{2x}\dfrac{dv}{dx}+4v\,e^{2x}$ in (1), we get

$$\frac{d^2v}{dx^2}+\frac{2x+3}{x+2}\frac{dv}{dx}=\frac{x+1}{x+2}e^{-x}$$

or

$$\frac{dp}{dx}+\frac{2x+3}{x+2}p=\frac{x+1}{x+2}e^{-x}, \qquad \left(\text{where } p=\frac{dv}{dx}\right)$$

which is a linear equation with p as dependent variable.

$$\text{I.F.}=e^{\int\frac{2x+3}{x+2}dx}=e^{\int\left(2-\frac{1}{x+2}\right)dx}=e^{2x-\log(x+2)}=\frac{e^{2x}}{x+2}$$

$$\therefore \quad p\cdot\frac{e^{2x}}{x+2}=\int\frac{x+1}{x+2}e^{-x}\cdot\frac{e^{2x}}{x+2}dx+c_1$$

$$=\int\frac{x+1}{(x+2)^2}e^xdx+c_1=\int\left\{\frac{1}{x+2}-\frac{1}{(x+2)^2}\right\}e^x\,dx+c_1$$

$$=\frac{1}{x+2}e^x+\int e^x\cdot\frac{1}{(x+2)^2}dx-\int e^x\frac{1}{(x+2)^2}dx+c_1,$$

integrating the first integral by parts.

$$\therefore \quad p=\frac{dv}{dx}=e^{-x}+c_1e^{-2x}(x+2).$$

Integrating this, we get

$$v=-e^{-x}-\frac{1}{2}c_1e^{-2x}(x+2)-\frac{1}{4}c_1e^{-2x}+c_2$$

$$=-e^{-x}-\frac{1}{4}c_1(2x+5)e^{-2x}+c_2.$$

$\therefore$ The general solution of (1) is

$$y = v\,e^{2x} = -\,e^x - \frac{1}{4}c_1\,(2x+5) + c_2\,e^{2x}.$$

Example 2: *Solve* $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)\,y = 0$.

(Garhwal 2003; Gorakhpur 08; Meerut 10)

Solution: The given equation can be written in the standard form as

$$\frac{d^2y}{dx^2} - \left(2 - \frac{1}{x}\right)\frac{dy}{dx} + \left(1 - \frac{1}{x}\right)y = 0. \qquad \ldots(1)$$

Here $1 + P + Q = 0$,

$\therefore$ $y = e^x$ is a part of the C.F. of the solution of (1).

Putting $y = v\,e^x, \dfrac{dy}{dx} = \dfrac{dv}{dx}e^x + v\,e^x$, and $\dfrac{d^2y}{dx^2} = \dfrac{d^2v}{dx^2}e^x + 2\dfrac{dv}{dx}e^x + v\,e^x$ in (1), we get

$$\frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} = 0 \Rightarrow \frac{dp}{dx} + \frac{p}{x} = 0\text{, where } p = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dp}{p} = -\frac{dx}{x} \Rightarrow \log p = -\log x + \log c_1$$

$$\Rightarrow \quad p = \frac{dv}{dx} = \frac{c_1}{x} \Rightarrow v = c_1 \log x + c_2.$$

$\therefore$ The complete solution of (1) is

$$y = v\,e^x = (c_1 \log x + c_2)\,e^x.$$

Example 3: *Solve* $x^2\dfrac{d^2y}{dx^2} - (x^2 + 2x)\dfrac{dy}{dx} + (x+2)\,y = x^3e^x$.

(Bundelkhand 2001; Garhwal 02;
Gorakhpur 06; Purvanchal 07; Kumaun 09; Meerut 2010B)

Solution: The standard form of the given equation is

$$\frac{d^2y}{dx^2} - \left(1 + \frac{2}{x}\right)\frac{dy}{dx} + \left(\frac{1}{x} + \frac{2}{x^2}\right)y = xe^x. \qquad \ldots(1)$$

Here $P + Qx = -\left(1 + \dfrac{2}{x}\right) + \left(\dfrac{1}{x} + \dfrac{2}{x^2}\right)x = 0$,

$\therefore$ $y = x$ is a part of the C.F. of the solution of (1).

Putting $y = vx, \dfrac{dy}{dx} = x\dfrac{dv}{dx} + v,\ \dfrac{d^2y}{dx^2} = x\dfrac{d^2v}{dx^2} + 2\dfrac{dv}{dx}$ in (1), we get

$$\frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x \quad \text{or} \quad \frac{dp}{dx} - p = e^x, \qquad \left(\text{where } p = \frac{dv}{dx}\right)$$

which is linear in *p*.

$$\therefore \quad \text{I.F.} = e^{\int -dx} = e^{-x}.$$

$$\therefore \quad pe^{-x} = \int e^x \cdot e^{-x}\,dx + c_1 = x + c_1.$$

$$\therefore \quad p = \frac{dv}{dx} = xe^x + c_1\,e^x.$$

Integrating this, we get

$$v = xe^x - e^x + c_1 e^x + c_2 .$$

$\therefore$ The complete solution of (1) is

$$y = v\,x = x^2\,e^x - xe^x + c_1\,xe^x + c_2 x .$$

Example 4: *Solve* $\dfrac{d^2y}{dx^2} - \cot x\,\dfrac{dy}{dx} - (1 - \cot x)\,y = e^x \sin x .$

(Meerut 2007B)

Solution: The given differential equation is

$$\frac{d^2y}{dx^2} - \cot x\,\frac{dy}{dx} - (1 - \cot x)\,y = e^x \sin x .$$

Here $1 + P + Q = 0$,

$\therefore$ $y = e^x$ is a part of the C.F. of (1).

Putting $y = ve^x$, $\dfrac{dy}{dx} = \dfrac{dv}{dx}\,e^x + ve^x$, and $\dfrac{d^2y}{dx^2} = \dfrac{d^2v}{dx^2}\,e^x + 2\,\dfrac{dv}{dx}\,e^x + ve^x$ in (1), we get

$$\frac{d^2v}{dx^2} + (2 - \cot x)\,\frac{dv}{dx} = \sin x$$

or
$$\frac{dp}{dx} + (2 - \cot x)\,p = \sin x, \qquad \left(\text{where } p = \frac{dv}{dx}\right)$$

which is linear in p.

$$\text{I. F.} = e^{\int (2 - \cot x)\,dx} = e^{2x - \log \sin x} = \frac{e^{2x}}{\sin x} .$$

$$\therefore \quad p\,\frac{e^{2x}}{\sin x} = \int \frac{e^{2x}}{\sin x}\,\sin x\,dx + c_1 = \frac{1}{2}\,e^{2x} + c_1 .$$

$$\therefore \quad p = \frac{dv}{dx} = \frac{1}{2}\sin x + c_1\,e^{-2x}\,\sin x .$$

Integrating this, we get

$$v = -\frac{1}{2}\cos x + \frac{c_1}{5}\,e^{-2x}\,(-2\sin x - \cos x) + c_2 .$$

$\therefore$ The complete solution of (1) is

$$y = v\,e^x = -\frac{1}{2}\,e^x \cos x - \frac{c_1}{5}\,e^{-x}\,(2\sin x + \cos x) + c_2\,e^x .$$

Example 5: *Solve* $x^2\,\dfrac{d^2y}{dx^2} - 2x\,(1 + x)\,\dfrac{dy}{dx} + 2\,(1 + x)\,y = x^3 .$

(Rohilkhand 2001; Garhwal 02, 07, 14; Kumaun 08, 10; Gorakhpur 10; Lucknow 05, 06)

Solution: The given equation can be written in the standard form as

$$\frac{d^2y}{dx^2} - 2\left(\frac{1}{x} + 1\right)\frac{dy}{dx} + 2\left(\frac{1}{x^2} + \frac{1}{x}\right)y = x . \qquad \ldots(1)$$

Here $P + Q\,x = 0$.

$\therefore$ $y = x$ is a part of the C.F. of (1).

Putting $y = v\,x$, $\dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$ and $\dfrac{d^2y}{dx^2} = x\dfrac{d^2v}{dx^2} + 2\dfrac{dv}{dx}$ in (1), we have

$$\frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1 \quad \text{or} \quad \frac{dp}{dx} - 2p = 1, \qquad \left(\text{where } p = \frac{dv}{dx}\right)$$

which is linear in p.

$$\text{I.F.} = e^{\int -2\,dx} = e^{-2x}.$$

$$\therefore \qquad pe^{-2x} = \int 1 \cdot e^{-2x}\,dx + c_1 = -\frac{1}{2}e^{-2x} + c_1.$$

$$\therefore \qquad p = \frac{dv}{dx} = -\frac{1}{2} + c_1\,e^{2x}.$$

Integrating, we get $v = -\dfrac{1}{2}x + \dfrac{c_1}{2}e^{2x} + c_2$.

$\therefore$ The complete solution of (1) is

$$y = v\,x = -\frac{1}{2}x^2 + \frac{c_1}{2}xe^{2x} + c_2x.$$

Example 6: *Solve* $\sin^2 x\dfrac{d^2y}{dx^2} = 2y$, *given* $y = \cot x$ *is a solution.*

(Purvanchal 2010; Lucknow 08; Kanpur 07, 08)

Solution: Putting $y = v\cot x$, $\dfrac{dy}{dx} = \dfrac{dv}{dx}\cot x - v\,\text{cosec}^2 x$,

$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}\cot x - 2\frac{dv}{dx}\text{cosec}^2 x + 2v\,\text{cosec}^2 x\cot x,$$

the given differential equation becomes

$$\sin^2 x\cot x\frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 0 \qquad \text{or} \qquad \frac{d^2v}{dx^2} - \frac{2}{\sin x\cos x}\frac{dv}{dx} = 0$$

or $$\frac{dp}{dx} - \frac{2}{\sin x\cos x}p = 0, \text{where } p = \frac{dv}{dx}$$

or $$\frac{dp}{p} = \frac{2}{\sin x\cos x}dx = 4\,\text{cosec}\,2x\,dx.$$

Integrating, we get

$$\log p = 4\cdot\frac{1}{2}\log\tan x + \log c_1$$

or $$p = \frac{dv}{dx} = c_1\tan^2 x = c_1(\sec^2 x - 1).$$

$$\therefore \qquad v = c_1\int(\sec^2 x - 1)\,dx + c_2 = c_1(\tan x - x) + c_2.$$

$\therefore$ the complete solution of the given equation is

$$y = v\cot x = c_1(1 - x\cot x) + c_2\cot x.$$

Example 7: *Solve* $(x\sin x + \cos x)\dfrac{d^2y}{dx^2} - x\cos x\dfrac{dy}{dx} + y\cos x = \sin x(x\sin x + \cos x)^2$.

Solution: The given equation in the standard form is

$$\frac{d^2 y}{dx^2} - \frac{x \cos x}{x \sin x + \cos x} \frac{dy}{dx} + \frac{\cos x}{x \sin x + \cos x} y = \sin x\,(x \sin x + \cos x) . \qquad \ldots(1)$$

Here $P + Qx = 0$,

$\therefore$ $y = x$ is a part of the C.F. of the solution of (1).

Putting $y = vx$ and the corresponding values of $\frac{dy}{dx}, \frac{d^2 y}{dx^2}$ in (1), we get

$$\frac{d^2 v}{dx^2} + \left(\frac{2}{x} - \frac{x \cos x}{x \sin x + \cos x}\right) \frac{dv}{dx} = \frac{\sin x\,(x \sin x + \cos x)}{x}$$

or
$$\frac{dp}{dx} + \left(\frac{2}{x} - \frac{x \cos x}{x \sin x + \cos x}\right) p = \frac{\sin x\,(x \sin x + \cos x)}{x} \cdot$$

which is linear in p where $p = \frac{dv}{dx} \cdot$

$\therefore$
$$\text{I. F.} = e^{\int \left(\frac{2}{x} - \frac{x \cos x}{x \sin x + \cos x}\right) dx}$$
$$= e^{2 \log x - \log (x \sin x + \cos x)} = \frac{x^2}{x \sin x + \cos x} \cdot$$

$\therefore$
$$p \cdot \frac{x^2}{x \sin x + \cos x} = \int x \sin x \, dx + c_1 = - x \cos x + \sin x + c_1 .$$

$\therefore$
$$p = \frac{dv}{dx} = \frac{1}{x^2} (- x \cos x + \sin x)\,(x \sin x + \cos x) + \frac{c_1}{x^2} (x \sin x + \cos x)$$

or
$$\frac{dv}{dx} = - \sin x \cos x - \frac{1}{x} \cos 2x + \frac{1}{x^2} \sin x \cos x + c_1 \left(\frac{1}{x} \sin x + \frac{1}{x^2} \cos x\right)$$

or
$$v = - \frac{1}{2} \int \sin 2x \, dx - \frac{1}{2} \int \left(\frac{2}{x} \cos 2x - \frac{1}{x^2} \sin 2x\right) dx + c_1 \int \left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right) dx + c_2$$
$$= \frac{1}{4} \cos 2x - \frac{1}{2} \int \frac{d}{dx}\left(\frac{\sin 2x}{x}\right) dx - c_1 \int \frac{d}{dx}\left(\frac{\cos x}{x}\right) dx + c_2$$
$$= \frac{1}{4} \cos 2x - \frac{1}{2} \frac{\sin 2x}{x} - c_1 \frac{\cos x}{x} + c_2 .$$

$\therefore$ The complete solution of (1) is

$$y = vx = \frac{1}{4} x \cos 2x - \frac{1}{2} \sin 2x - c_1 \cos x + c_2 x .$$

Example 8: *Solve* $x \frac{dy}{dx} - y = (x - 1) \left(\frac{d^2 y}{dx^2} - x + 1\right)$.

(Garhwal 2001)

Solution: The given equation in the standard form is

$$\frac{d^2 y}{dx^2} - \frac{x}{x - 1} \frac{dy}{dx} + \frac{y}{x - 1} = x - 1. \qquad \ldots(1)$$

Here $P + Qx = 0$.

$\therefore$ $y = x$ is a part of the C.F. of the solution of (1).

Putting $y = vx$ and the corresponding values of $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ in (1), we get

$$\frac{d^2 v}{dx^2} + \left(\frac{2}{x} - \frac{x}{x-1}\right)\frac{dv}{dx} = \frac{x-1}{x}$$

or $$\frac{dp}{dx} + \left(\frac{2}{x} - \frac{x}{x-1}\right) p = \frac{x-1}{x}, \qquad \left(\text{where } p = \frac{dv}{dx}\right)$$

which is linear in p.

$\therefore$ $$\text{I.F.} = e^{\int\left(\frac{2}{x} - \frac{x}{x-1}\right)dx} = e^{\int\left(\frac{2}{x} - 1 - \frac{1}{x-1}\right)dx} = e^{2\log x - x - \log(x-1)} = \frac{x^2}{x-1} e^{-x}.$$

$\therefore$ $$p \cdot \frac{x^2}{(x-1)} e^{-x} = \int \frac{x-1}{x} \cdot \frac{x^2}{x-1} e^{-x} dx + c_1$$

$$= \int xe^{-x}\, dx + c_1 = -xe^{-x} - e^{-x} + c_1 .$$

$\therefore$ $$p = \frac{dv}{dx} = -\frac{x-1}{x} - \frac{(x-1)}{x^2} + \frac{c_1 (x-1)}{x^2} e^x$$

or $$\frac{dv}{dx} = -1 + \frac{1}{x^2} + c_1 \left(\frac{1}{x} - \frac{1}{x^2}\right) e^x.$$

Integrating, we get

$$v = -x - \frac{1}{x} + c_1 \frac{1}{x} e^x + c_2 .$$

$\therefore$ The complete solution of (1) is

$$y = vx = -x^2 - 1 + c_1 e^x + c_2 x$$

or $$y = c_1 e^x + c_2 x - (1 + x^2) .$$

Comprehensive Exercise 1

Solve the following differential equations :

1. $(x+1)\frac{d^2 y}{dx^2} - 2(x+3)\frac{dy}{dx} + (x+5)\, y = e^x.$ **(Kumaun 2012)**

2. $x\frac{d^2 y}{dx^2} - 2(x+1)\frac{dy}{dx} + (x+2)\, y = (x-2)\, e^x.$

3. $x\frac{d^2 y}{dx^2} - (2x+1)\frac{dy}{dx} + (x+1)\, y = (x^2 + x - 1)\, e^{2x}.$

4. $\frac{d^2 y}{dx^2} + (1 - \cot x)\frac{dy}{dx} - y \cot x = \sin^2 x .$

5. $(x - x^2)\dfrac{d^2y}{dx^2} - (1 - 2x)\dfrac{dy}{dx} + (1 - 3x + x^2)\, y = (1 - x)^3.$

6. $(3 - x)\dfrac{d^2y}{dx^2} - (9 - 4x)\dfrac{dy}{dx} + (6 - 3x)\, y = 0\,.$

7. $x\dfrac{d^2y}{dx^2} + (x - 2)\dfrac{dy}{dx} - 2y = x^3.$ **(Kanpur 2011)**

8. $(2x - 1)\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + (3 - 2x)\, y = 2e^x.$

9. $x\dfrac{d^2y}{dx^2} + (1 - x)\dfrac{dy}{dx} - y = e^x.$ **(Kumaun 2009)**

10. Solve $x^2 y_2 + xy_1 - 9y = 0$, given that $y = x^3$ is a solution.

11. Solve $\dfrac{d^2y}{dx^2} + \left(1 + \dfrac{2}{x}\cot x - \dfrac{2}{x^2}\right) y = x\cos x\,,$ **(Gorakhpur 2008, 11)**

given that $\dfrac{\sin x}{x}$ is an integral included in the C.F.

12. Solve $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - a^2 y = 0$, given that $y = ce^{a\sin^{-1}x}$ is an integral.

Answers 1

1. $y = -\dfrac{1}{4}\, x\, e^x + \dfrac{1}{5}\, c_1 e^x (x + 1)^5 + c_2 e^x$

2. $y = -\dfrac{1}{2}\, x^2 e^x + xe^x + \dfrac{1}{3}\, c_1 x^3 e^x + c_2 e^x$

3. $y = x\, e^{2x} + c_1 x^2 e^x + c_2 e^x$

4. $y = -\dfrac{1}{10}(\sin 2x - 2\cos 2x) + c_1(\sin x - \cos x) + c_2 e^{-x}$

5. $y = -x + \dfrac{1}{2}\, c_1 x^2 e^{-x} + c_2 e^x$

6. $y = c_1 e^{3x}(4x^3 - 42x^2 + 150x - 183) + c_2 e^x$

7. $y = x^3 + (c_1 - 3)\, x^2 - 2(c_1 - 3)(x - 1) + c_2\, e^{-x}$

8. $y = -c_1 x e^{-x} - xe^x + e^x \displaystyle\int (2x - 1)\log(2x - 1)\, dx$

$$- 2e^x \int [e^{-2x}(2x - 1)\int e^{2x} \log(2x - 1)\, dx]\, dx + c_2\, e^x$$

9. $y = e^x \log x + c_1\, e^x \displaystyle\int \frac{e^{-x}}{x}\, dx + c_2\, e^x$

10. $y = c_1 x^{-3} + c_2 x^3$

11. $y = \dfrac{x^2 \sin x}{6} + c_1 \left[- x \cos x + 2 \sin x \log \sin x - \dfrac{2 \sin x}{x} \int \log \sin x \, dx \right] + c_2 \dfrac{\sin x}{x}$

12. $y = \left[- \dfrac{c_1}{2a} e^{-2a \sin^{-1} x} + c_2 \right] \cdot e^{a \sin^{-1} x}$

4 Removal of the First Derivative (Reduction to Normal Form)

(Agra 2007; Gorakhpur 05)

If we are unable to obtain a part of the C.F. of the solution of the differential equation

$$\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R, \qquad \ldots(1)$$

then we cannot solve (1) by the method given in article 2. In such cases the equation (1) can sometimes be solved by reducing it into the form in which the term containing the first derivative is absent. For this we first change the dependent variable from y to v in the equation (1) by putting $y = uv$, where u is some function of x.

Then $\qquad \dfrac{dy}{dx} = v \dfrac{du}{dx} + u \dfrac{dv}{dx}, \; \dfrac{d^2 y}{dx^2} = v \dfrac{d^2 u}{dx^2} + 2 \dfrac{dv}{dx} \cdot \dfrac{du}{dx} + u \dfrac{d^2 v}{dx^2} \cdot$

Putting the values of $y, \dfrac{dy}{dx}, \dfrac{d^2 y}{dx^2}$ in (1), we get

$$\left[v \frac{d^2 u}{dx^2} + 2 \frac{dv}{dx} \cdot \frac{du}{dx} + u \frac{d^2 v}{dx^2} \right] + P \left[v \frac{du}{dx} + u \frac{dv}{dx} \right] + Q \, vu = R$$

or $$u \frac{d^2 v}{dx^2} + u \left[P + \frac{2}{u} \frac{du}{dx} \right] \frac{dv}{dx} + v \left[\frac{d^2 u}{dx^2} + P \frac{du}{dx} + Qu \right] = R. \qquad \ldots(2)$$

To remove the term of the first derivative $\dfrac{dv}{dx}$ we choose u such that

$$P + \frac{2}{u} \frac{du}{dx} = 0 \qquad \text{or} \qquad \frac{du}{u} = - \frac{1}{2} P \, dx$$

or $$\log u = - \frac{1}{2} \int P \, dx \qquad \text{or} \qquad u = e^{-\frac{1}{2} \int P \, dx} \qquad \ldots(3)$$

Now the equation (2) becomes

$$\frac{d^2 v}{dx^2} + \frac{v}{u} \left[\frac{d^2 u}{dx^2} + P \frac{du}{dx} + Qu \right] = \frac{R}{u} \cdot \qquad \ldots(4)$$

But from (3), $\dfrac{du}{dx} = - \dfrac{1}{2} Pu,$

$$\frac{d^2 u}{dx^2} = - \frac{1}{2} \left[P \frac{du}{dx} + u \frac{dP}{dx} \right] = - \frac{1}{2} \left[P \left(- \frac{1}{2} Pu \right) + u \frac{dP}{dx} \right]$$

$$= \frac{1}{4} P^2 \, u - \frac{1}{2} u \frac{dP}{dx} \cdot$$

Putting these values in (4), we get

$$\frac{d^2 v}{dx^2} + v \left[\frac{1}{4} P^2 - \frac{1}{2} \frac{dP}{dx} - P \cdot \frac{1}{2} P + Q \right] = \frac{R}{e^{-\frac{1}{2} \int P \, dx}}$$

or $$\frac{d^2 v}{dx^2} + v \left[Q - \frac{P^2}{4} - \frac{1}{2} \frac{dP}{dx} \right] = R \, e^{\frac{1}{2} \int P \, dx} \qquad \text{or} \qquad \frac{d^2 v}{dx^2} + Xv = Y \, ,$$

where $$X = Q - \frac{1}{2} \frac{dP}{dx} - \frac{1}{4} P^2 \text{ and } Y = Re^{\frac{1}{2} \int P \, dx}.$$

The reduced equation (5) may easily be integrated. The equation (5) is called the **normal form** of the equation (1).

Note: The students should remember the values of u, X and Y. They can write the reduced equation (5) directly.

Illustrative Examples

Example 9: *Solve* $\frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + (4x^2 - 1) \, y = -3 \, e^{x^2} \sin 2x$.

(Rohilkhand 2001; Meerut 05; Agra 07; Gorakhpur 09; Garhwal 10B, 15; Kumaun 06, 13)

Solution: Here $P = -4x, Q = 4x^2 - 1, R = -3e^{x^2} \sin 2x$.

We choose $u = e^{-\frac{1}{2} \int P \, dx} = e^{-\frac{1}{2} \int -4x \, dx} = e^{x^2}$.

Putting $y = uv$ in the given equation, it reduces to its normal form

$$\frac{d^2 v}{dx^2} + Xv = Y, \qquad \text{...(1)}$$

where $$X = Q - \frac{1}{2} \frac{dP}{dx} - \frac{1}{4} P^2 = 4x^2 - 1 - \frac{1}{2} (-4) - \frac{1}{4} \cdot 16x^2 = 1,$$

and $$Y = Re^{\frac{1}{2} \int P \, dx} = -3e^{x^2} \sin 2x \cdot e^{-x^2} = -3 \sin 2x.$$

Hence the normal form (1) of the given equation is

$$\frac{d^2 v}{dx^2} + v = -3 \sin 2x \qquad \text{or} \qquad (D^2 + 1) \, v = -3 \sin 2x. \qquad \text{...(2)}$$

Now (2) is a linear differential equation with constant coefficients.

A.E. is $m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i$.

$\therefore$ C.F. of the solution of (2) $= c_1 \cos x + c_2 \sin x$.

$$\text{P.I.} = \frac{1}{D^2 + 1} (-3 \sin 2x) = \frac{-3}{-2^2 + 1} \sin 2x = \sin 2x.$$

$\therefore$ the solution of (2) is $v = c_1 \cos x + c_2 \sin x + \sin 2x$.

Hence the general solution of the given equation is

$$y = uv = e^{x^2}\ (c_1 \cos x + c_2 \sin x + \sin 2x)\ .$$

Example 10: *Solve* $\dfrac{d^2 y}{dx^2} - 2 \tan x \dfrac{dy}{dx} + 5y = \sec x \,.e^x.$

(Agra 2006; Gorakhpur 06; Kanpur 09; Garhwal 10)

Solution: Here $P = -2 \tan x, Q = 5, R = e^x \sec x$.

We choose $\quad u = e^{-\frac{1}{2}\int P\, dx} = e^{\int \tan x\, dx} = e^{\log\ \sec x} = \sec x.$

Substituting $y = uv$ in the given equation, it reduces to its normal form

$$\frac{d^2 v}{dx^2} + Xv = Y\ , \qquad \ldots(1)$$

where $\quad X = Q - \dfrac{1}{2}\dfrac{dP}{dx} - \dfrac{1}{4}P^2 = 5 - \dfrac{1}{2}(-2\sec^2 x) - \tan^2 x = 6$

and $\quad Y = Re^{\frac{1}{2}\int P\, dx} = e^x \sec x\, (\sec x)^{-1} = e^x.$

Hence the equation (1) is

$$\frac{d^2 v}{dx^2} + 6v = e^x \qquad \text{or} \qquad (D^2 + 6)\, v = e^x\ . \qquad \ldots(2)$$

A.E. is $m^2 + 6 = 0 \Rightarrow m = \pm \sqrt{6}i$.

$\therefore$ C.F. of the solution of (2) $= c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x.$

$$\text{P.I.} = \frac{1}{D^2 + 6} e^x = \frac{1}{7} e^x.$$

$\therefore$ the solution of (2) is $v = c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x + \dfrac{1}{7} e^x$.

Hence the complete solution of the given equation is

$$y = uv = \sec x\, (c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x + \frac{1}{7} e^x)\ .$$

Example 11: *Solve* $\dfrac{d^2 y}{dx^2} + \dfrac{1}{x^{1/3}}\dfrac{dy}{dx} + \left(\dfrac{1}{4x^{2/3}} - \dfrac{1}{6x^{4/3}} - \dfrac{6}{x^2}\right) y = 0\ .$

Solution: Here $P = x^{-1/3}, Q = \dfrac{1}{4x^{2/3}} - \dfrac{1}{6x^{4/3}} - \dfrac{6}{x^2}, R = 0$.

We choose $u = e^{-\frac{1}{2}\int P\, dx} = e^{-\frac{1}{2}\int x^{-1/3} dx} = e^{-\frac{3}{4}x^{2/3}}$.

Substituting $y = uv$ in the given equation, it reduces to its normal form

$$\frac{d^2 v}{dx^2} + Xv = Y\ , \qquad \ldots(1)$$

where $\quad X = Q - \dfrac{1}{2}\dfrac{dP}{dx} - \dfrac{1}{4}P^2$

$$= \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2} - \frac{1}{2}\left(-\frac{1}{3}x^{-4/3}\right) - \frac{1}{4}x^{-2/3} = -\frac{6}{x^2},$$

and $$Y = Re^{\frac{1}{2}\int P\,dx} = 0.$$

Hence the normal form (1) of the given equation is

$$\frac{d^2v}{dx^2} - \frac{6}{x^2}v = 0 \text{ or } x^2\frac{d^2v}{dx^2} - 6v = 0, \qquad \ldots(2)$$

which is a homogeneous linear equation.

In order to solve it, putting $x = e^z$, the differential equation (2) becomes

$$[D(D-1) - 6]\,v = 0, \text{ where } D \equiv d/dz$$

or $$(D^2 - D - 6)\,v = 0.$$

A.E. is $m^2 - m - 6 = 0$, giving $m = -2, 3$.

$\therefore$ the solution of (2) is $v = c_1e^{-2z} + c_2e^{3z} = c_1x^{-2} + c_2x^3$.

Hence the complete solution of the given equation is

$$y = uv = e^{(-3/4)x^{2/3}}(c_1x^{-2} + c_2x^3).$$

Example 12: *Solve* $\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + (4x^2 - 3)\,y = e^{x^2}$.

(Gorakhpur 2005, 07; Lucknow 09; Garhwal 11; Kumaun 12)

Solution: Here $P = -4x, Q = 4x^2 - 3, R = e^{x^2}$.

We choose $u_1 = e^{-\frac{1}{2}\int P\,dx} = e^{\int 2x\,dx} = e^{x^2}$.

Substituting $y = uv$ in the given equation, it reduces to its normal form

$$\frac{d^2v}{dx^2} + Xv = Y, \qquad \ldots(1)$$

where $$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = -1$$

and $$Y = Re^{\frac{1}{2}\int P\,dx} = 1.$$

Hence the normal form (1) of the given equation is

$$\frac{d^2v}{dx^2} - v = 1 \text{ or } (D^2 - 1)\,v = 1. \qquad \ldots(2)$$

A.E. is $m^2 - 1 = 0$, giving $m = \pm 1$.

$\therefore$ C.F. of the solution of (2) $= c_1e^x + c_2e^{-x}$.

$$\text{P.I.} = \frac{1}{D^2 - 1}(1) = \frac{1}{D^2 - 1}e^{0x} = \frac{1}{0^2 - 1}e^{0x} = -1.$$

$\therefore$ the solution of (2) is $v = c_1e^x + c_2e^{-x} - 1$.

Hence the complete solution of the given equation is

$$y = uv = e^{x^2}(c_1e^x + c_2e^{-x} - 1).$$

Example 13: *Solve* $\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+(x^2+1)\,y=x^3+3x$.

Solution: Here $P=2x, Q=x^2+1, R=x\,(x^2+3)$.

We choose $u=e^{-\frac{1}{2}\int P\,dx}=e^{-x^2/2}$.

Substituting $y=uv$ in the given equation, it reduces to its normal form

$$\frac{d^2v}{dx^2}+Xv=Y\,. \qquad \ldots(1)$$

where $\quad X=Q-\dfrac{1}{2}\dfrac{dP}{dx}-\dfrac{1}{4}P^2=0$

and $\quad Y=R\,e^{\frac{1}{2}\int P\,dx}=x(x^2+3)\,e^{x^2/2}$.

Hence the normal form (1) of the given equation is

$$\frac{d^2v}{dx^2}=x\,(x^2+3)\,e^{x^2/2}. \qquad \ldots(2)$$

Integrating (2), we get

$$\frac{dv}{dx}=\int x^3\,e^{x^2/2}dx+3\int x\,e^{x^2/2}dx+c_1$$

$$=\int x^2\left(x\,e^{x^2/2}\right)dx+3\int x\,e^{x^2/2}dx+c_1$$

$$=x^2e^{x^2/2}-2\int x\,e^{x^2/2}dx+3\int x\,e^{x^2/2}dx+c_1,$$

integrating the first integral by parts

$$=x^2e^{x^2/2}+e^{x^2/2}+c_1\,.$$

Integrating again, we get

$$v=\int x^2e^{x^2/2}dx+\int e^{x^2/2}dx+c_1x+c_2=x\,e^{x^2/2}+c_1x+c_2.$$

Hence the complete solution of the given equation is

$$y=uv=x+(c_1x+c_2)\,e^{-x^2/2}.$$

Example 14: *Solve* $\dfrac{d^2y}{dx^2}-\dfrac{1}{x^{1/2}}\dfrac{dy}{dx}+\dfrac{1}{4x^2}(x+x^{1/2}-8)\,y=0$.

Solution: Here $P=-\dfrac{1}{x^{1/2}}, Q=\dfrac{1}{4x^2}(x+x^{1/2}-8),\ R=0$.

We choose $u=e^{-\frac{1}{2}\int P\,dx}=e^{\frac{1}{2}\int x^{-1/2}dx}=e^{x^{1/2}}=e^{\sqrt{x}}$.

Substituting $y=uv$ in the given equation, it reduces to its normal form

$$\frac{d^2v}{dx^2}+Xv=Y, \qquad \ldots(1)$$

where $\quad X=Q-\dfrac{1}{2}\dfrac{dP}{dx}-\dfrac{1}{4}P^2=-\dfrac{2}{x^2}$

and $$Y = R e^{\frac{1}{2}\int P\,dx} = 0.$$

Hence the normal form (1) of the given equation is

$$\frac{d^2 v}{dx^2} + v\left(-\frac{2}{x^2}\right) = 0 \quad \text{or} \quad x^2 \frac{d^2 v}{dx^2} - 2v = 0. \qquad \ldots(2)$$

This is a homogeneous linear equation.

Putting $x = e^z$, the differential equation (2) becomes

$$\{D(D-1) - 2\}\, v = 0 \quad \text{or} \quad (D^2 - D - 2)\, v = 0,$$

where $D \equiv d/dz$.

A.E. is $m^2 - m - 2 = 0$ or $(m-2)(m+1) = 0$.

$\therefore \quad m = 2, -1.$

Thus the solution of (2) is $v = c_1 e^{2z} + c_2 e^{-z} = c_1 x^2 + c_2 x^{-1}$.

Hence the complete solution of the given equation is

$$y = uv = e^{\sqrt{x}} (c_1 x^2 + c_2 x^{-1}).$$

Comprehensive Exercise 2

Solve the following equations by removing the first derivative :

1. $x^2 \dfrac{d^2 y}{dx^2} - 2(x^2 + x) \dfrac{dy}{dx} + (x^2 + 2x + 2)\, y = 0$.

(Garhwal 2000; Bundelkhand 01; Kanpur 08)

2. $\dfrac{d^2 y}{dx^2} - 4x \dfrac{dy}{dx} + (4x^2 - 1)\, y = -3e^{x^2} (\sin 2x + 5e^{-2x} + 6)$.

3. $\dfrac{d^2 y}{dx^2} - \dfrac{2}{x} \dfrac{dy}{dx} + \left(1 + \dfrac{2}{x^2}\right) y = x e^x$.

(Kanpur 2009; Garhwal 14)

4. $\dfrac{d}{dx}\left(\cos^2 x \dfrac{dy}{dx}\right) + y \cos^2 x = 0$.

(Rohilkhand 2007)

5. $\dfrac{d^2 y}{dx^2} + \dfrac{2}{x} \dfrac{dy}{dx} + n^2 y = 0$.

(Agra 2008)

6. $x \dfrac{d}{dx}\left(x \dfrac{dy}{dx} - y\right) - 2x \dfrac{dy}{dx} + 2y + x^2 y = 0$.

(Rohilkhand 2008)

7. $\dfrac{d^2 y}{dx^2} - \dfrac{2}{x} \dfrac{dy}{dx} + \left(a^2 + \dfrac{2}{x^2}\right) y = 0$.

8. $\left(\dfrac{d^2 y}{dx^2} + y\right) \cot x + 2\left(\dfrac{dy}{dx} + y \tan x\right) = \sec x$.

(Garhwal 2008)

9. $x^2 (\log x)^2 \dfrac{d^2 y}{dx^2} - 2x \log x \dfrac{dy}{dx} + [2 + \log x - 2(\log x)^2]\, y = x^2 (\log x)^3$.

10. $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = 0$. **(Agra 2008)**

Answers 2

1. $y = xe^x (c_1 x + c_2)$
2. $y = e^{x^2} (c_1 \cos x + c_2 \sin x + \sin 2x - 3e^{-2x} - 18)$
3. $y = x(c_1 \cos x + c_2 \sin x + \frac{1}{2} e^x)$
4. $y = (c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x) \sec x$
5. $y = \frac{1}{x} c_1 \cos(nx + c_2)$ or $y = \frac{1}{x}(c_1 e^{nx} + c_2 e^{-nx})$
6. $y = x(c_1 \cos x + c_2 \sin x)$
7. $y = x c_1 \cos(ax + c_2)$ or $y = x(c_1 \cos ax + c_2 \sin ax)$
8. $y = \frac{1}{2} \sin x + (c_1 x + c_2) \cos x$
9. $y = \{c_1 x^2 + c_2 x^{-1} + \frac{1}{3} x^2 \log x\} . \log x$
10. $y = (\sec x) . \{c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x\}$

5 Transformation of the Equation by Changing the Independent Variable

Sometimes the equation may become easily integrable by changing the independent variable.

Let the linear equation of second order be

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R. \qquad \ldots(1)$$

Let the independent variable be changed from x to z with the help of a relation of the form $z = f(x)$.

Then $$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}, \quad \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz} \cdot \frac{d^2z}{dx^2}.$$

Substituting these values in (1), we get

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz} \cdot \frac{d^2z}{dx^2} + P\left[\frac{dy}{dz} \cdot \frac{dz}{dx}\right] + Qy = R$$

or $$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right) + Qy = R$$

or $$\frac{d^2y}{dz^2}+\frac{\dfrac{d^2z}{dx^2}+P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}\cdot\frac{dy}{dz}+\frac{Q}{\left(\dfrac{dz}{dx}\right)^2}y=\frac{R}{\left(\dfrac{dz}{dx}\right)^2}$$

or $$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1, \qquad \ldots(2)$$

where $$P_1=\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)\Big/\left(\frac{dz}{dx}\right)^2, Q_1=Q\Big/\left(\frac{dz}{dx}\right)^2 \text{ and } R_1=R\Big/\left(\frac{dz}{dx}\right)^2.$$

Here P_1, Q_1 and R_1 are functions of x and may be expressed as functions of z with the help of the relation $z=f(x)$.

How to choose z? We would like to choose z in such a way that the equation (2) can be easily integrated.

Case I: Let us choose z in such a way that P_1 vanishes *i.e.*,

$$\frac{d^2z}{dx^2}+P\frac{dz}{dx}=0 \quad\Rightarrow\quad \frac{d}{dx}\left(\frac{dz}{dx}\right)+P\frac{dz}{dx}=0$$

$$\Rightarrow \quad \frac{dz}{dx}=e^{-\int P\,dx}\Rightarrow z=\int[e^{-\int P\,dx}]\,dx.$$

Then the equation (2) is reduced to $\dfrac{d^2y}{dz^2}+Q_1y=R_1$.

Now this equation can be easily solved, if

(i) Q_1 is a constant because then it is a linear equation with constant coefficients,

(ii) Q_1 is of the form (constant)/z^2 because then it is a linear homogeneous equation with variable coefficients.

Case II: Suppose we choose z such that

$$Q_1=\frac{Q}{\left(\dfrac{dz}{dx}\right)^2}=\pm a^2 \text{ (i.e., a constant)}$$

or $$a\frac{dz}{dx}=\sqrt{(\pm Q)} \quad\text{or}\quad az=\int\sqrt{(\pm Q)}\,dx,$$

(+ ive or – ive sign is taken to make the expression under the radical sign + ive).

With this choice of z, the equation (2) is reduced to

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+a^2y=R_1.$$

This equation can be easily solved if P_1 comes out to be a constant because then it is a linear equation with constant coefficients.

Note 1: Students should remember the values of P_1, Q_1 and R_1 in equation (2).

Note 2: There are only two choices for z, either $P_1=0$ or $Q_1=a^2$. Sometimes it is possible to make both the choices to get the solution of the given equation.

Illustrative Examples

Example 15: *Solve* $x\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-4x^3y=8x^3\sin x^2$.

(Meerut 2004, 13B; Garhwal 09; Agra 05; Gorakhpur 09; Kumaun 08, 11)

Solution: The given equation written in the standard form is

$$\frac{d^2y}{dx^2}-\frac{1}{x}\frac{dy}{dx}-4x^2y=8x^2\sin x^2. \qquad \ldots(1)$$

Here $P=-1/x,\ Q=-4x^2,\ R=8x^2\sin x^2$.

Changing the independent variable from x to z by a relation of the form $z=f(x)$, the given equation is transformed into

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1, \qquad \ldots(2)$$

where $$P_1=\frac{\dfrac{d^2z}{dx^2}+P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2},\ Q_1=\frac{Q}{\left(\dfrac{dz}{dx}\right)^2},R_1=\frac{R}{\left(\dfrac{dz}{dx}\right)^2}\cdot$$

Choosing z such that $Q_1=\dfrac{-4x^2}{(dz/dx)^2}=\text{constant}=-1$ (say), we have

$$(dz/dx)^2=4x^2. \qquad \therefore \qquad dz/dx=2x\Rightarrow z=x^2.$$

Now $$P_1=\frac{2-\dfrac{1}{x}\cdot 2x}{4x^2}=0,\ R_1=2\sin x^2=2\sin z.$$

$\therefore$ The transformed equation (2) is

$$\frac{d^2y}{dz^2}-y=2\sin z \quad\text{or}\quad (D^2-1)\,y=2\sin z. \qquad \ldots(3)$$

A.E. is $m^2-1=0\Rightarrow m^2=1\Rightarrow m=\pm 1$.

$\therefore$ C.F. of the solution of (3) $=c_1e^z+c_2e^{-z}$.

$$\text{P.I.}=\frac{1}{D^2-1}2\sin z=\frac{2}{-1-1}\sin z=-\sin z.$$

$\therefore$ Solution of the equation (3) is

$$y=c_1e^z+c_2e^{-z}-\sin z.$$

Hence the complete solution of the equation (1) is

$$y=c_1e^{x^2}+c_2e^{-x^2}-\sin x^2.$$

Example 16: *Solve* $\dfrac{d^2y}{dx^2}+\cot x\dfrac{dy}{dx}+4y\,\mathrm{cosec}^2x=0$.

(Meerut 2001; Agra 06; Gorakhpur 05)

Solution: Here $P = \cot x, Q = 4 \operatorname{cosec}^2 x, R = 0$.

Changing the independent variable from x to z by a relation of the form $z = f(x)$, the given equation is transformed into

$$\frac{d^2 y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1, \qquad \ldots(1)$$

where $$P_1 = \frac{\dfrac{d^2 z}{dx^2} + P \dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}, Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2}, R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2} \cdot$$

Choosing z such that $Q_1 = \dfrac{4 \operatorname{cosec}^2 x}{(dz/dx)^2} = \text{constant} = 1$ (say), we have

$$(dz/dx)^2 = 4 \operatorname{cosec}^2 x$$

or $$\frac{dz}{dx} = 2 \operatorname{cosec} x \quad \text{or} \quad z = 2 \log \tan \frac{x}{2} \cdot$$

Now $$P_1 = \frac{-2 \operatorname{cosec} x \cot x + 2 \operatorname{cosec} x \cot x}{4 \operatorname{cosec}^2 x} = 0, R_1 = 0.$$

$\therefore$ the transformed equation (1) is

$$\frac{d^2 y}{dz^2} + y = 0 \quad \text{or} \quad (D^2 + 1) y = 0. \qquad \ldots(2)$$

A.E. is $m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i$.

$\therefore$ Solution of the equation (2) is

$$y = c_1 \cos z + c_2 \sin z .$$

Hence the complete solution of the given equation is

$$y = c_1 \cos\left(2 \log \tan \frac{x}{2}\right) + c_2 \sin\left(2 \log \tan \frac{x}{2}\right).$$

Example 17: *Solve* $\cos x \dfrac{d^2 y}{dx^2} + \sin x \dfrac{dy}{dx} - 2 y \cos^3 x = 2 \cos^5 x$.

(Meerut 2006; Rohilkhand 07; Agra 07; Avadh 11; Purvanchal 06, 10)

Solution: The given equation can be written in the standard form as

$$\frac{d^2 y}{dx^2} + \tan x \frac{dy}{dx} - 2 y \cos^2 x = 2 \cos^4 x. \qquad \ldots(1)$$

Here $P = \tan x, Q = -2 \cos^2 x, R = 2 \cos^4 x$.

Changing the independent variable from x to z by a relation of the form $z = f(x)$, the given equation is transformed into

$$\frac{d^2 y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1, \qquad \ldots(2)$$

where $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}\cdot$$

Let us choose z such that $Q_1 = \dfrac{-2\cos^2 x}{(dz/dx)^2}$ = constant = – 2 (say).

Then $\left(\dfrac{dz}{dx}\right)^2 = \cos^2 x$ or $\dfrac{dz}{dx} = \cos x$ or $z = \sin x$.

Now $P_1 = 0, R_1 = 2\cos^2 x = 2(1 - \sin^2 x) = 2(1 - z^2)$.

∴ The transformed equation (2) is

$$\frac{d^2y}{dz^2} - 2y = 2(1 - z^2). \qquad \text{...(3)}$$

A.E. is $m^2 - 2 = 0$, giving $m = \pm\sqrt{2}$.

∴ C.F. of the solution of (3) $= c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z}$.

$$\text{P.I.} = \frac{1}{D^2 - 2} 2(1 - z^2) = -\left(1 - \frac{D^2}{2}\right)^{-1}(1 - z^2)$$

$$= -\left(1 + \frac{1}{2}D^2 + \dots\right)(1 - z^2) = -(1 - z^2) + \frac{1}{2}\cdot 2 = z^2.$$

∴ The solution of the equation (3) is

$$y = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z} + z^2.$$

Hence the complete solution of the given equation is

$$y = c_1 e^{\sqrt{2}\sin x} + c_2 e^{-\sqrt{2}\sin x} + \sin^2 x.$$

Example 18: *Solve* $x^6 \dfrac{d^2y}{dx^2} + 3x^5 \dfrac{dy}{dx} + a^2 y = \dfrac{1}{x^2}\cdot$ **(Garhwal 2001)**

Solution: The given equation can be written in the standard form as

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{a^2}{x^6}y = \frac{1}{x^8}\cdot$$

Here $P = \dfrac{3}{x}, Q = \dfrac{a^2}{x^6}, R = \dfrac{1}{x^8}\cdot$

To solve this differential equation we change the independent variable from x to z by choosing z such that

$$\frac{Q}{(dz/dx)^2} = \frac{a^2/x^6}{(dz/dx)^2} = \text{constant} = a^2 \text{ (say)}.$$

Then $\left(\dfrac{dz}{dx}\right)^2 = \dfrac{1}{x^6}$ or $\dfrac{dz}{dx} = \dfrac{1}{x^3}$ or $z = -1/(2x^2)$.

Now by the substitution $z = -1/(2x^2)$, the given differential equation is transformed into

$$\frac{d^2 y}{dz^2} + P_1 \frac{dy}{dz} + Q_1\, y = R_1, \text{ where}$$

$$P_1 = \frac{\dfrac{d^2 z}{dx^2} + P\dfrac{dz}{dx}}{(dz/dx)^2} = \frac{(-3/x^4) + (3/x)(1/x^3)}{(dz/dx)^2} = 0,$$

$$Q_1 = \frac{Q}{(dz/dx)^2} = a^2, R_1 = \frac{R}{(dz/dx)^2} = \frac{1/x^8}{1/x^6} = \frac{1}{x^2} = -2z.$$

$\therefore$ the transformed equation is

$$\frac{d^2 y}{dz^2} + a^2 y = -2z \quad \text{or} \quad (D^2 + a^2)\, y = -2z. \qquad \ldots(1)$$

The C.F. of the solution of (1) $= c_1 \cos az + c_2 \sin az$.

$$\text{P.I.} = \frac{1}{D^2 + a^2}(-2z) = \frac{1}{a^2}\left(1 + \frac{D^2}{a^2}\right)^{-1}(-2z)$$

$$= \frac{1}{a^2}\left(1 - \frac{1}{a^2}D^2 + \ldots\right)(-2z) = \frac{1}{a^2}(-2z).$$

$\therefore$ the solution of the equation (1) is

$$y = c_1 \cos az + c_2 \sin az - \frac{1}{a^2}\cdot 2z.$$

Hence the complete solution of the given equation is

$$y = c_1 \cos \frac{a}{2x^2} - c_2 \sin \frac{a}{2x^2} + \frac{1}{a^2 x^2}$$

or
$$y = c_1 \cos \frac{a}{2x^2} + c_2 \sin \frac{a}{2x^2} + \frac{1}{a^2 x^2}.$$

Example 19: *Solve* $\dfrac{d^2 y}{dx^2} + (\tan x - 1)^2 \dfrac{dy}{dx} - n(n-1)\, y \sec^4 x = 0$.

Solution: Comparing the given differential equation with the standard form

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = R, \text{ we have}$$

$$P = (\tan x - 1)^2, Q = -n(n-1)\sec^4 x, R = 0.$$

To solve the given differential equation we change the independent variable from x to z by choosing z such that

$$\frac{Q}{(dz/dx)^2} = \frac{-n(n-1)\sec^4 x}{(dz/dx)^2} = \text{constant} = -n(n-1), \text{ say}.$$

Then $\left(\dfrac{dz}{dx}\right)^2 = \sec^4 x \quad \text{or} \quad \dfrac{dz}{dx} = \sec^2 x \quad \text{or} \quad z = \tan x.$

Now by the substitution $z = \tan x$, the given differential equation is transformed into

$$\frac{d^2 y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1, \text{ where}$$

$$P_1 = \frac{\dfrac{d^2 z}{dx^2} + P \dfrac{dz}{dx}}{(dz/dx)^2} = \frac{2 \sec^2 x \tan x + (\tan x - 1)^2 \sec^2 x}{\sec^4 x} = 1,$$

$$Q_1 = \frac{Q}{(dz/dx)^2} = -n(n-1), \quad R_1 = \frac{R}{(dz/dx)^2} = 0.$$

$\therefore$ the transformed equation is

$$\frac{d^2 y}{dz^2} + \frac{dy}{dz} - n(n-1)\, y = 0 \quad \text{or} \quad \{D^2 + D - n(n-1)\}\, y = 0. \qquad \ldots(1)$$

A.E. is $m^2 + m - n(n-1) = 0$ or $(m+n)\{m - (n-1)\} = 0.$

$\therefore$ $m = -n, n-1.$

$\therefore$ the solution of the equation (1) is

$$y = c_1 e^{-nz} + c_2 e^{(n-1)z}.$$

Hence the complete solution of the given equation is

$$y = c_1 e^{-n \tan x} + c_2 e^{(n-1) \tan x}.$$

Comprehensive Exercise 3

Solve the following equations :

1. $\dfrac{d^2 y}{dx^2} + (3 \sin x - \cot x) \dfrac{dy}{dx} + 2 y \sin^2 x = e^{-\cos x} \sin^2 x.$

2. $\dfrac{d^2 y}{dx^2} - (1 + 4e^x) \dfrac{dy}{dx} + 3e^{2x} y = e^{2(x + e^x)}.$ **(Gorakhpur 2008, 11)**

3. $\dfrac{d^2 y}{dx^2} + \left(1 - \dfrac{1}{x}\right) \dfrac{dy}{dx} + 4x^2\, y e^{-2x} = 4(x^2 + x^3)\, e^{-3x}.$ **(Rohilkhand 2006)**

4. $(a^2 - x^2) \dfrac{d^2 y}{dx^2} - \dfrac{a^2}{x} \dfrac{dy}{dx} + \dfrac{x^2}{a}\, y = 0.$ **(Bundelkhand 2001)**

5. $\dfrac{d^2 y}{dx^2} - \dfrac{1}{x} \dfrac{dy}{dx} + 4x^2 y = x^4.$

6. $x \dfrac{d^2 y}{dx^2} + (4x^2 - 1) \dfrac{dy}{dx} + 4x^3 y = 2x^3.$

7. $(1 + x)^2 \dfrac{d^2 y}{dx^2} + (1 + x) \dfrac{dy}{dx} + y = 4 \cos \log (1 + x).$ **(Gorakhpur 2010)**

8. $\dfrac{d^2 y}{dx^2} - \cot x \dfrac{dy}{dx} - y \sin^2 x = 0.$

9. $x^4 \dfrac{d^2 y}{dx^2} + 2x^3 \dfrac{dy}{dx} + n^2 y = 0$. **(Kumaun 2010)**

10. $(1 + x^2)^2 \dfrac{d^2 y}{dx^2} + 2x (1 + x^2) \dfrac{dy}{dx} + 4 y = 0$.

Answers 3

1. $y = c_1 e^{\cos x} + c_2 e^{2 \cos x} + \dfrac{1}{6} e^{-\cos x}$
2. $y = c_1 e^{3e^x} + c_2 e^{e^x} - e^{2e^x}$
3. $y = c_1 \cos \{2e^{-x} (1 + x)\} - c_2 \sin \{2e^{-x} (1 + x)\} + e^{-x} (1 + x)$
4. $y = c_1 \cos \left\{\dfrac{\sqrt{(a^2 - x^2)}}{\sqrt{a}}\right\} + c_2 \sin \left\{\dfrac{\sqrt{(a^2 - x^2)}}{\sqrt{a}}\right\}$
5. $y = c_1 \cos (x^2 + c_2) + \dfrac{1}{4} x^2$
6. $y = (c_1 + c_2 x^2) e^{-x^2} + \dfrac{1}{2}$
7. $y = c_1 \cos \{\log (1 + x) + c_2\} + 2 \log (1 + x) \cdot \sin \log (1 + x)$
8. $y = c_1 e^{-\cos x} + c_2 e^{\cos x} - \cos x$
9. $y = c_1 \cos \left(\dfrac{n}{x} + c_2\right)$
10. $y = c_1 \cos (2 \tan^{-1} x) + c_2 \sin (2 \tan^{-1} x)$, or $y (1 + x^2) = c_1 (1 - x^2) + 2 c_2 x$.

6 Method of Variation of Parameters

Now we shall discuss an elegant but somewhat artificial method for finding the complete primitive of a linear equation whose complementary function is known.

Let the linear equation of second order be

$$\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Q y = R. \qquad \text{...(1)}$$

Suppose $y = Au + Bv$ is the complementary function of (1), where A and B are arbitrary constants and u and v are functions of x. Then $y = Au + Bv$ is the solution of

$$\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = 0.$$

$\therefore$ $(Au_2 + Bv_2) + P (Au_1 + Bv_1) + Q (Au + Bv) = 0$

or $A (u_2 + Pu_1 + Qu) + B (v_2 + Pv_1 + Qv) = 0$

$\Rightarrow$ $u_2 + Pu_1 + Qu = 0$ and $v_2 + Pv_1 + Qv = 0$. ...(2)

Now, suppose that

$$y = Au + Bv \qquad \ldots(3)$$

is the complete primitive of (1) where A and B are not constants but functions of x, so chosen that (1) will be satisfied.

Differentiating (3),

$$\frac{dy}{dx} = A_1 u + B_1 v + Au_1 + Bv_1.$$

Let us choose A and B such that

$$A_1 u + B_1 v = 0. \qquad \ldots(4)$$

Then $\quad \dfrac{dy}{dx} = Au_1 + Bv_1 \quad$ and $\quad \dfrac{d^2 y}{dx^2} = Au_2 + Bv_2 + A_1 u_1 + B_1 v_1.$

Putting these values in (1), we get

$$(Au_2 + Bv_2 + A_1 u_1 + B_1 v_1) + P\,(Au_1 + Bv_1) + Q\,(Au + Bv) = R$$

or $\quad A\,(u_2 + Pu_1 + Qu) + B\,(v_2 + Pv_1 + Qv) + A_1 u_1 + B_1 v_1 = R$

or $\quad A_1 u_1 + B_1 v_1 = R_1, \qquad$ [using 2] ...(5)

Solving (4) and (5), we get

$$A_1 = \frac{vR}{u_1 v - uv_1} \text{ and } B_1 = \frac{uR}{v_1 u - vu_1}, \text{ which on integration give } A \text{ and } B.$$

$$\therefore \quad A = \int \frac{v\,R\,dx}{u_1 v - uv_1} + c_1 \text{ and } B = \int \frac{u\,R\,dx}{v_1 u - vu_1} + c_2,$$

where c_1 and c_2 are parameters.

Substituting these values of A and B in (3), we get the complete primitive of (1).

Note : Since the general solution is obtained by varying the arbitrary constants of the complementary function, therefore the method is known as *method of variation of parameters*.

7 Wronskian Method

Alternate method to find C.S:

Consider the linear equation of second order

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = R. \qquad \ldots(1)$$

The P.I. of (1) is given by

$$\text{P.I.} = -u\int \frac{vR}{W}\,dx + v\int \frac{uR}{W}\,dx = -u\int \frac{vR}{uv_1 - vu_1}\,dx + v\int \frac{uR}{uv_1 - vu_1}\,dx, \quad \ldots(2)$$

where u and v are defined by the C.F. of

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad \ldots(3)$$

i.e., by $y = c_1\,u + c_2 v$, where c_1 and c_2 are arbitrary constants and $W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix}$ is called the *Wronskian* of u and v, also written as $W(u\ v)$.

$\therefore$ C.S. = C.F. + P.I.

Note : The method of variation of parameters must be used, if instructed to do so in the question.

Illustrative Examples

Example 20: *Solve by the method of variation of parameters*

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x.$$

(Rohilkhand 2001, 11; Lucknow 09; Garhwal 05; Kumaun 06, 13; Avadh 05; Meerut 04B)

Solution: The C.F. of the given equation *i.e.*, the solution of the equation

$$\frac{d^2y}{dx^2} + 4y = 0 \text{ is}$$

$$y = c_1 \cos 2x + c_2 \sin 2x, \text{ where } c_1 \text{ and } c_2 \text{ are constants.}$$

Let $\quad y = A\cos 2x + B\sin 2x \quad$ ...(1)

be the complete primitive of the given equation where A and B are functions of x, so chosen that the given equation will be satisfied.

Then $\quad \dfrac{dy}{dx} = -2A\sin 2x + 2B\cos 2x + \dfrac{dA}{dx}\cos 2x + \dfrac{dB}{dx}\sin 2x.$

Let us choose A and B such that

$$\frac{dA}{dx}\cos 2x + \frac{dB}{dx}\sin 2x = 0. \quad \text{...(2)}$$

Then $\quad \dfrac{dy}{dx} = -2A\sin 2x + 2B\cos 2x$

and $\quad \dfrac{d^2y}{dx^2} = -2\dfrac{dA}{dx}\sin 2x + 2\dfrac{dB}{dx}\cos 2x - 4A\cos 2x - 4B\sin 2x.$

Putting these values in the given equation, we get

$$-2\sin 2x\frac{dA}{dx} + 2\cos 2x\frac{dB}{dx} = 4\tan 2x$$

or $\quad -\sin 2x\dfrac{dA}{dx} + \cos 2x\dfrac{dB}{dx} = 2\tan 2x. \quad$ ...(3)

Solving (2) and (3), we get

$$\frac{dA}{dx} = -\frac{2\sin^2 2x}{\cos 2x}, \quad \frac{dB}{dx} = 2\sin 2x.$$

Integrating these, we get

$$A = -2\int \frac{(1-\cos^2 2x)}{\cos 2x}\,dx + c_1 = -2\int (\sec 2x - \cos 2x)\,dx + c_1$$

$$= -\log(\sec 2x + \tan 2x) + \sin 2x + c_1$$

and $\qquad B = -\cos 2x + c_2.$

Putting the values of A and B in (1), the complete primitive of the given equation is

$$y = c_1 \cos 2x + c_2 \sin 2x - [\log(\sec 2x + \tan 2x)] \cdot \cos 2x.$$

Alternate method to find P.I.:

Here $u = \cos 2x, v = \sin 2x, R = 4 \tan 2x$

and $\qquad W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2.$

Thus, $\qquad \text{P.I.} = -u \int \frac{vR}{W} dx + v \int \frac{uR}{W} dx$

$$= -\cos 2x \int 2 \sin 2x \tan 2x \, dx + \sin 2x \int 2 \cos 2x \tan 2x \, dx$$

$$= -2 \cos 2x \int (\sec 2x - \cos 2x) \, dx + 2 \sin 2x \int \sin 2x \, dx$$

$$= -\cos 2x [\log(\sec 2x + \tan 2x) - \sin 2x] - \sin 2x \cos 2x.$$

$$= -\cos 2x \log(\sec 2x + \tan 2x).$$

Hence, the C.S. is $y = c_1 \cos 2x + c_2 \sin 2x - (\cos 2x) \log(\sec 2x + \tan 2x)$.

Example 21: *(i) Solve by the method of variation of parameters*

$$\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + y = e^x \log x.$$

Solution: First we shall find the C.F. of the given equation *i.e.*, the solution of the equation

$$\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + y = 0 \qquad \text{or} \qquad (D^2 - 2D + 1)\, y = 0.$$

$$\text{A.E. is } m^2 - 2m + 1 = 0 \qquad \text{or} \qquad (m-1)^2 = 0, \text{ giving } m = 1, 1.$$

$\therefore \qquad$ C.F. is $y = (c_1 + c_2 x)\, e^x$.

Now assume A and B as the functions of x in such a way that the given equation is satisfied by

$$y = (A + Bx)\, e^x = Ae^x + Bxe^x. \qquad \ldots(1)$$

$$\therefore \qquad \frac{dy}{dx} = Ae^x + B(e^x + xe^x) + e^x \frac{dA}{dx} + xe^x \frac{dB}{dx}$$

$$= Ae^x + B(e^x + xe^x), \text{ assuming } e^x \frac{dA}{dx} + xe^x \frac{dB}{dx} = 0. \qquad \ldots(2)$$

$$\therefore \qquad \frac{d^2 y}{dx^2} = Ae^x + B(2e^x + xe^x) + e^x \frac{dA}{dx} + (e^x + xe^x)\frac{dB}{dx}.$$

Now putting these values in the given equation, we get

$$e^x \frac{dA}{dx} + e^x (1 + x)\frac{dB}{dx} = e^x \log x. \qquad \ldots(3)$$

Solving (2) and (3), we get

$$\frac{dA}{dx} = -x \log x \text{ and } \frac{dB}{dx} = \log x .$$

Integrating these, we get

$$A = -\int x \log x \, dx + c_1 = -\left(\frac{x^2}{2} \log x - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx \right) + c_1$$

$$= -\left(\frac{x^2}{2} \log x - \frac{x^2}{4} \right) + c_1,$$

$$B = \int \log x \, dx + c_2 = x \log x - \int \frac{1}{x} \cdot x \, dx + c_2 = x \log x - x + c_2 .$$

Putting these values of A and B in (1), the general solution of the given equation is

$$y = -e^x \left(\frac{x^2}{2} \log x - \frac{x^2}{4} + c_1 \right) + x e^x (x \log x - x + c_2)$$

$$= (c_1 + c_2 x) e^x + \frac{1}{4} x^2 e^x (2 \log x - 3) .$$

Alternate method to find P.I.

Here $u = e^x, v = x e^x, R = e^x \log x$

and $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x) e^x \end{vmatrix} = e^{2x}.$$

Thus $$\text{P.I.} = -u \int \frac{uR}{W} dx + v \int \frac{uR}{W} dx$$

$$= -e^x \int \frac{xe^x \cdot e^x \log x}{e^{2x}} dx + xe^x \int \frac{e^x \cdot e^x \log x}{e^{2x}} dx$$

$$= -e^x \int x \log x \, dx + xe^x \int \log x \, dx$$

$$= -e^x \left(\frac{x^2}{2} \log x - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx \right) + x \cdot e^x \left(x \log x - \int \frac{1}{x} \cdot x \, dx \right)$$

$$= -e^x \left(\frac{x^2}{2} \log x - \frac{x^2}{4} \right) + x e^x (x \log x - x) = \frac{1}{4} x^2 e^x (2 \log x - 3) .$$

Hence, the C.S. is $y = (c_1 + c_2 x) e^x + \frac{1}{4} x^2 e^x (2 \log x - 3)$.

Example 22: *(ii) Solve by the method of variation of parameters*

$$\frac{d^2 y}{dx^2} + n^2 y = \sec nx .$$

(Meerut 2003, 07; Gorakhpur 11; Garhwal 07, 11; Kumaun 09, 12; Lucknow 10; Avadh 09, 10, 13)

Solution: The C.F. of the given equation *i.e.* the solution of the equation

$$\frac{d^2 y}{dx^2} + n^2 y = 0 \text{ is}$$

$y = c_1 \cos nx + c_2 \sin nx$, where c_1 and c_2 are constants.

Let $y = A\cos nx + B\sin nx$ ...(1)

be the general solution of the given equation where A and B are functions of x, so chosen that the given equation will be satisfied.

Then $\dfrac{dy}{dx} = -An\sin nx + Bn\cos nx + \dfrac{dA}{dx}\cos nx + \dfrac{dB}{dx}\sin nx$.

Let us choose A and B such that

$$\frac{dA}{dx}\cos nx + \frac{dB}{dx}\sin nx = 0 \qquad ...(2)$$

Then $\dfrac{dy}{dx} = -An\sin nx + Bn\cos nx$

and $\dfrac{d^2y}{dx^2} = -An^2\cos nx - Bn^2\sin nx - n\dfrac{dA}{dx}\sin nx + n\dfrac{dB}{dx}\cos nx$.

Putting these values in the given equation, we get

$$-n\frac{dA}{dx}\sin nx + n\frac{dB}{dx}\cos nx = \sec nx. \qquad ...(3)$$

Solving (2) and (3), we get

$$\frac{dA}{dx} = -\frac{1}{n}\tan nx \quad \text{and} \quad \frac{dB}{dx} = \frac{1}{n}\cdot$$

Integrating these, we get

$$A = \frac{1}{n^2}\log\cos nx + c_1,\ B = \frac{x}{n} + c_2.$$

Putting these values of A and B in (1), the general solution of the given equation is

$$y = c_1\cos nx + c_2\sin nx + \frac{1}{n^2}\cos nx\log\cos nx + \frac{x}{n}\sin nx.$$

Example 23: *Apply the method of variation of parameters to solve* $\dfrac{d^2y}{dx^2} + y = \tan x$.

Solution: The given differential equation is

$$\frac{d^2y}{dx^2} + y = \tan x \quad \text{or} \quad (D^2+1)\,y = \tan x \qquad ...(1)$$

A.E. is $m^2 = -1$ or $m = \pm i$

$\therefore$ C.F. is $y = A\cos x + B\sin x$

Here $u = \cos x, v = \sin x, R = \tan x$

and $W = uv_1 - u_1 v = \cos^2 x + \sin^2 x = 1$

$\therefore$ $\text{P.I.} = -\displaystyle\int \frac{vR}{W}dx + v\int \frac{uR}{W}dx.$

We have $\displaystyle\int \frac{-vR}{W}dx = -\int \frac{\sin x\tan x}{1}dx = -\int \frac{\sin^2 x}{\cos x}dx = -\int \frac{1-\cos^2 x}{\cos x}dx$

$$= \int (\cos x - \sec x)\,dx = \sin x - \log(\sec x + \tan x)$$

and $$\int \frac{uR}{W}\, dx = \int \frac{\cos x \cdot \tan x}{1} = \int \sin x \, dx = -\cos x$$

$\therefore$ $$\text{P.I.} = [\sin x - \log(\sec x + \tan x)] \cos x - \cos x \sin x$$
$$= -\cos x \log(\sec x + \tan x)$$

Hence the complete solution is $y = A \cos x + B \sin x - \cos x \log(\sec x + \tan x)$.

Example 24: *Solve by the method of variation of parameters*

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x^2 e^x.$$

(Rohilkhand 2006, 11; Purvanchal 06; Meerut 07; Garhwal 01, 10; Gorakhpur 06, 10; Lucknow 08, 11; Avadh 14; Kanpur 15)

Solution: First we shall find the C.F. of the given equation *i.e.*, the solution of the equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 0 . \qquad \ldots(1)$$

This is a homogeneous linear equation. Putting $x = e^z$, the differential equation (1) becomes

$$\{D(D-1) + D - 1\}\, y = 0 \text{, where } D \equiv \frac{d}{dz}$$

or $$(D^2 - 1)\, y = 0$$

whose solution is $y = c_1 e^z + c_2 e^{-z}$.

$\therefore$ solution of (1) is $y = c_1 x + c_2 \dfrac{1}{x} \cdot$

Let $$y = Ax + \frac{B}{x} \qquad \ldots(2)$$

be the complete primitive of the given equation where A and B are functions of x, so chosen that the given equation will be satisfied.

Then $$\frac{dy}{dx} = A - \frac{B}{x^2} + \frac{dA}{dx} x + \frac{1}{x} \cdot \frac{dB}{dx} \cdot$$

Let us choose A and B such that

$$x \frac{dA}{dx} + \frac{1}{x} \cdot \frac{dB}{dx} = 0 . \qquad \ldots(3)$$

Then $$\frac{dy}{dx} = A - \frac{B}{x^2} \quad \text{and} \quad \frac{d^2 y}{dx^2} = \frac{dA}{dx} - \frac{1}{x^2} \frac{dB}{dx} + \frac{2}{x^3} B .$$

Putting these values in the given equation, we get

$$\frac{dA}{dx} - \frac{1}{x^2} \frac{dB}{dx} = e^x . \qquad \ldots(4)$$

Solving (3) and (4), we get

$$\frac{dA}{dx} = \frac{1}{2} e^x \text{ and } \frac{dB}{dx} = -\frac{x^2}{2} e^x.$$

Integrating these, we get

$$A = \frac{1}{2}e^x + c_1 \quad \text{and} \quad B = -\frac{1}{2}x^2e^x + xe^x - e^x + c_2.$$

Putting the values of A and B in (2), the complete primitive of the given equation is

$$y = c_1 x + \frac{c_2}{x} + e^x - \frac{1}{x}e^x.$$

Example 25: *Solve by the method of variation of parameters*

$$(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (1-x)^2.$$

(Kumaun 2011)

Solution: First we shall find the C.F. of the given equation *i.e.*, the solution of the equation

$$(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

or $$\frac{d^2y}{dx^2} + \frac{x}{1-x}\frac{dy}{dx} - \frac{y}{1-x} = 0. \qquad \ldots(1)$$

Here $P + Qx = 0$.

$\therefore$ $y = x$ is a part of the C.F.

Putting $y = vx$ in (1), the reduced equation is

$$\frac{d^2v}{dx^2} + \left(\frac{x}{1-x} + \frac{2}{x}\right)\frac{dv}{dx} = 0$$

or $$\frac{dp}{dx} + \left(\frac{x}{1-x} + \frac{2}{x}\right)p = 0, \text{ where } p = \frac{dv}{dx}$$

or $$\frac{dp}{dx} + \left(-1 - \frac{1}{x-1} + \frac{2}{x}\right)p = 0 \quad \text{or} \quad \frac{dp}{p} = \left(1 + \frac{1}{x-1} - \frac{2}{x}\right)dx.$$

Integrating, $\log p = x + \log(x-1) - 2\log x + \log c_1$

or $$p = \frac{dv}{dx} = \frac{c_1(x-1)e^x}{x^2} = c_1\left(\frac{1}{x} - \frac{1}{x^2}\right)e^x.$$

Integrating again, we get

$$v = c_1\int\left(\frac{1}{x} - \frac{1}{x^2}\right)e^x\,dx + c_2 \quad \text{or} \quad v = \frac{c_1}{x}e^x + c_2.$$

$\therefore$ solution of (1) is $y = vx = c_1e^x + c_2x$.

Let $$y = Ae^x + Bx \qquad \ldots(2)$$

be the complete primitive of the given equation where A and B are functions of x, so chosen that the given equation will be satisfied.

Then $$\frac{dy}{dx} = Ae^x + B + e^x\frac{dA}{dx} + x\frac{dB}{dx}.$$

Let us choose A and B such that

$$e^x \frac{dA}{dx} + x \frac{dB}{dx} = 0 . \qquad \ldots(3)$$

Then $\dfrac{dy}{dx} = Ae^x + B$ and $\dfrac{d^2 y}{dx^2} = e^x \dfrac{dA}{dx} + \dfrac{dB}{dx} + e^x A$.

Putting these values in the given equation, we get

$$e^x \frac{dA}{dx} + \frac{dB}{dx} = 1 - x. \qquad \ldots(4)$$

Solving (3) and (4), we get

$$\frac{dA}{dx} = - xe^{-x} \quad \text{and} \quad \frac{dB}{dx} = 1.$$

Integrating these, we get

$$A = (1 + x)\, e^{-x} + c_1 \text{ and } B = x + c_2.$$

Putting the values of A and B in (2) the general solution of the given equation is

$$y = c_1 e^x + c_2 x + 1 + x + x^2.$$

Example 26: *Solve by the method of variation of parameters*

$$\frac{d^2 y}{dx^2} + (1 - \cot x) \frac{dy}{dx} - y \cot x = \sin^2 x.$$

(Kumaun 2008; Garhwal 12)

Solution: First we shall find the C.F. of the given equation *i.e.*, the solution of the equation

$$\frac{d^2 y}{dx^2} + (1 - \cot x) \frac{dy}{dx} - y \cot x = 0 . \qquad \ldots(1)$$

Here $1 - P + Q = 0$, $\therefore$ $y = e^{-x}$ is a part of the C.F.

Putting $y = ve^{-x}$, the equation (1) reduces to

$$\frac{d^2 v}{dx^2} - (1 + \cot x) \frac{dv}{dx} = 0 \quad \text{or} \quad \frac{dp}{dx} - (1 + \cot x)\, p = 0 \text{ , where } p = \frac{dv}{dx}$$

or $\dfrac{dp}{p} = (1 + \cot x)\, dx$.

Integrating, $\log p = x + \log \sin x + \log c_1$.

$\therefore$ $p = \dfrac{dv}{dx} = c_1 e^x \sin x$

or $v = c_1 \displaystyle\int e^x \sin x \, dx + c_2 = c_1 \cdot \frac{1}{2} e^x (\sin x - \cos x) + c_2$.

$\therefore$ the solution of (1) *i.e.*, the C.F. of the given equation is

$$y = ve^{-x} = c_1 \cdot \frac{1}{2} (\sin x - \cos x) + c_2 e^{-x}.$$

Let $y = A (\sin A - \cos x) + Be^{-x}$ ...(2)

be the complete solution of the given equation where A and B are functions of x, so chosen that the given equation will be satisfied.

$\therefore$ $\dfrac{dy}{dx} = A (\cos x + \sin x) - Be^{-x} + \dfrac{dA}{dx} (\sin x - \cos x) + \dfrac{dB}{dx} e^{-x}$.

Let us choose A and B such that

$$\frac{dA}{dx}(\sin x-\cos x)+\frac{dB}{dx}e^{-x}=0\,. \qquad \ldots(3)$$

$$\therefore \qquad \frac{dy}{dx}=A(\cos x+\sin x)-Be^{-x}$$

and

$$\frac{d^2y}{dx^2}=\frac{dA}{dx}(\cos x+\sin x)-\frac{dB}{dx}e^{-x}+A(-\sin x+\cos x)+Be^{-x}.$$

Putting these values in the given equation, we get

$$\frac{dA}{dx}(\cos x+\sin x)-\frac{dB}{dx}e^{-x}=\sin^2 x. \qquad \ldots(4)$$

Solving (3) and (4), we get

$$\frac{dA}{dx}=\frac{1}{2}\sin x \qquad \text{and} \qquad \frac{dB}{dx}=\frac{1}{2}e^x(\sin x\cos x-\sin^2 x)\,.$$

Integrating these, we get

$$A=-\frac{1}{2}\cos x+c_1$$

and

$$B=\frac{1}{4}\int e^x(\sin 2x-1+\cos 2x)\,dx+c_2$$

$$=\frac{1}{4}\cdot\frac{e^x}{5}(\sin 2x-2\cos 2x)-\frac{e^x}{4}+\frac{1}{4}\cdot\frac{e^x}{5}(\cos 2x+2\sin 2x)+c_2$$

$$=\frac{e^x}{20}(3\sin 2x-\cos 2x)-\frac{e^x}{4}+c_2.$$

Putting the values of A and B in (2), the general solution of the given equation is

$$y=c_1(\sin x-\cos x)+c_2e^{-x}-\frac{1}{10}(\sin 2x-2\cos 2x)\,.$$

Comprehensive Exercise 4

Solve the following differential equations by the method of variation of parameters:

1. $\dfrac{d^2y}{dx^2}+y=x\,.$ **(Agra 2007)**

2. $\dfrac{d^2y}{dx^2}+a^2y=\operatorname{cosec} ax.$ **(Kumaun 2010)**

3. $x^2\dfrac{d^2y}{dx^2}-2x(1+x)\dfrac{dy}{dx}+2(x+1)\,y=x^3.$ **(Gorakhpur 2007; Garhwal 13)**

4. $(1-x^2)\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}-(1+x^2)\,y=x\,.$ **(Garhwal 2002, 08)**

5. $(1-x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=x(1-x^2)^{3/2}.$

6. $(x+2)\dfrac{d^2y}{dx^2}-(2x+5)\dfrac{dy}{dx}+2y=(x+1)e^x$. **(Garhwal 2009)**

7. $\dfrac{d^2y}{dx^2}+y=\operatorname{cosec} x$. **(Purvanchal 2007, 10; Kumaun 14)**

8. $\dfrac{d^2y}{dx^2}-y=\dfrac{2}{1+e^x}$. **(Agra 2008; Gorakhpur 05; Purvanchal 09; Kanpur 07, 14; Kumaun 07, 15)**

9. $xy_1-y=(x-1)(y_2-x+1)$.

Answers 4

1. $y=c_1\cos x+c_2\sin x+x$
2. $y=c_1\cos ax+c_2\sin ax-\dfrac{x}{a}\cos ax+\dfrac{1}{a^2}\sin ax\log\sin ax$
3. $y=c_1x+c_2xe^{2x}-\dfrac{1}{2}x^2-\dfrac{1}{4}x$
4. $y=\dfrac{1}{(1-x^2)}\cdot(c_1\cos x+c_2\sin x+x)$
5. $y=c_1[\sqrt{(1-x^2)}+x\sin^{-1}x]+c_2x-(x/9)(1-x^2)^{3/2}$
6. $y=c_1(2x+5)+c_2e^{2x}-e^x$
7. $y=c_1\cos x+c_2\sin x-x\cos x+\sin x\log\sin x$
8. $y=c_1e^x+c_2e^{-x}+e^x\log\left(\dfrac{1+e^x}{e^x}\right)-1-e^{-x}\log(1+e^x)$
9. $y=c_1e^x+c_2x-(1+x^2)$

8 Method of Operational Factors

In some cases the method of operational factors easily solves the equation.

Let the linear equation of second order

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R,$$

be expressed in the form $f(D)y=R$

or $$f_1(D)f_2(D)y=R$$

i.e., $f(D)$ can be resolved into a product of two factors $f_1(D)$ and $f_2(D)$, such that, if $f_2(D)$ operates upon y, and then $f_1(D)$ operates upon the result of the operation, the same result is obtained as if $f(D)$ operates upon y.

Note: With the exception of the classes of equations, the factors of $f(D)$ are generally not commutative. Hence a great care is to be taken in writing them in the right order.

Illustrative Examples

Example 27: *Solve* $x\dfrac{d^2y}{dx^2}+(1-x)\dfrac{dy}{dx}-y=e^x$.

Solution: In the symbolic form, the given equation is

$$[xD^2+(1-x)\,D-1]\;y=e^x,\text{ where } D\equiv d/dx$$

or $\quad (xD+1)\,(D-1)\;y=e^x. \qquad \ldots(1)$

[Note that here the factors are not commutative because on expansion $\{(D-1)(xD+1)\}\;y$ gives $\{xD^2+(2-x)D-1\}\,y]$.

Let $\quad (D-1)\;y=v. \qquad \ldots(2)$

Then (1) gives $(xD+1)\,v=e^x$ or $x\dfrac{dv}{dx}+v=e^x$ or $\dfrac{dv}{dx}+\dfrac{1}{x}v=\dfrac{e^x}{x}$,

which is a linear differential equation of the first order.

Now $\quad$ I.F. $=e^{\int(1/x)\,dx}=e^{\log x}=x.$

$\therefore \quad vx=\displaystyle\int x\frac{e^x}{x}\,dx+c_1=e^x+c_1$ or $v=e^x\,x^{-1}+c_1x^{-1}$.

Putting the value of v in (2), we get

$$(D-1)\;y=e^xx^{-1}+c_1x^{-1}$$

or $\quad \dfrac{dy}{dx}-y=e^xx^{-1}+c_1x^{-1}$, which is again linear.

Here $\quad$ I.F. $=e^{\int -dx}=e^{-x}$.

$$\therefore \quad ye^{-x}=\int(e^xx^{-1}+c_1x^{-1})\,e^{-x}\,dx+c_2$$

$$=\int\frac{1}{x}\,dx+c_1\int\frac{e^{-x}}{x}\,dx+c_2=\log x+c_1\int\frac{e^{-x}}{x}\,dx+c_2.$$

$$\therefore \quad y=e^x\log x+c_1e^x\int\frac{e^{-x}}{x}\,dx+c_2e^x,$$

which is the required general solution of the given differential equation.

Example 28: *Solve* $3x^2\dfrac{d^2y}{dx^2}+(2+6x-6x^2)\dfrac{dy}{dx}-4y=0$.

Solution: In the symbolic form, the given equation is

$$[3x^2D^2+(2+6x-6x^2)\,D-4]\;y=0$$

or $\quad [(3x^2D^2+6x\,D+2D)-(6x^2\,D+4)]\;y=0$

or $\quad [D\,(3x^2D+2)-2\,(3x^2\,D+2)]\;y=0$

or $\quad (D-2)\,(3x^2D+2)\;y=0. \qquad \ldots(1)$

[Here factors are not commutative because on expansion

$$(3x^2 D+2)(D-2)\,y \text{ gives } \{3x^2 D^2+(2-6x^2)D-4\}\,y].$$

Let $(3x^2 D+2)\,y=v$. ...(2)

Then (1) gives $(D-2)\,v=0 \Rightarrow \dfrac{dv}{dx}-2v=0$

$$\frac{dv}{dx}=2v \quad \Rightarrow \quad v=c_1 e^{2x}.$$

Putting the value of v in (2), we get

$$(3x^2 D+2)\,y=c_1 e^{2x}$$

or $$3x^2\frac{dy}{dx}+2y=c_1e^{2x} \Rightarrow \frac{dy}{dx}+\frac{2}{3x^2}\,y=\frac{c_1}{3x^2}e^{2x}.$$

∴ $$\text{I.F.}=e^{\int \frac{2}{3x^2}dx}=e^{\frac{2}{3}\left(-\frac{1}{x}\right)}=e^{-\frac{2}{3x}}.$$

∴ $$ye^{-\frac{2}{3x}}=\frac{c_1}{3}\int x^{-2}e^{2x}\cdot e^{-\frac{2}{3x}}\,dx+c_2$$

or $$y=c_2\,e^{\frac{2}{3x}}+\frac{c_1}{3}e^{\frac{2}{3x}}\int x^{-2}e^{2x-\frac{2}{3x}}\,dx,$$

which is the required solution.

Example 29: *Solve* $x\dfrac{d^2y}{dx^2}-(x+2)\dfrac{dy}{dx}+2y=x^3$.

Solution: In the symbolic form, the given equation is

$$[xD^2-(x+2)D+2]\,y=x^3$$

or $(xD-2)(D-1)\,y=x^3$ ...(1)

[Here factors are not commutative because on expansion

$$\{(D-1)(xD-2)\}\,y \text{ gives } \{xD^2-(x+1)D+2\}\,y].$$

Let $(D-1)\,y=v$. ...(2)

Then (1) gives

$$(xD-2)\,v=x^3 \qquad \text{or} \qquad x\frac{dv}{dx}-2v=x^3$$

or $\dfrac{dv}{dx}-\dfrac{2}{x}v=x^2$, which is linear.

∴ $$\text{I.F.}=e^{\int(-2/x)dx}=e^{-2\log x}=\frac{1}{x^2}.$$

∴ $$v\cdot\frac{1}{x^2}=\int\left(x^2\cdot\frac{1}{x^2}\right)dx+c_1=x+c_1.$$

∴ $$v=x^3+c_1x^2.$$

Putting the value of v in (2), we get

$$(D-1)\ y = x^3 + c_1x^2 \quad \text{or} \quad \frac{dy}{dx} - y = x^3 + c_1x^2, \text{which is again linear.}$$

$$\therefore \qquad \text{I. F.} = e^{\int -dx} = e^{-x}.$$

$$\therefore \qquad ye^{-x} = \int e^{-x}\,(x^3 + c_1x^2)\,dx + c_2$$

$$= -(x^3 + c_1x^2)\,e^{-x} - (3x^2 + 2c_1x)\,e^{-x} - (6x + 2c_1)\,e^{-x} - 6e^{-x} + c_2$$

or
$$y = -(x^3 + c_1x^2) - (3x^2 + 2c_1x) - (6x + 2c_1) - 6 + c_2e^x$$

$$= -x^3 - (c_1 + 3)\,x^2 - 2\,(c_1 + 3)\,x - 2\,(c_1 + 3) + c_2\,e^x$$

or
$$y = -x^3 - (c_1 + 3)\,\{x^2 + 2x + 2\} + c_2e^x.$$

which is the required general solution of the given differential equation.

Example 30: *Solve* $[(x+3)\,D^2 - (2x+7)\,D + 2]\ y = (x+3)^2\,e^x$.

Solution: The given equation can be written as

$$\{(x+3)\,D - 1\}\,(D-2)\ y = (x+3)^2\,e^x. \qquad \text{...(1)}$$

[Here factors are not commutative because on expansion

$(D-2)\,\{(x+3)\,D - 1\}\ y$ gives $\{(x+3)\,D^2 - (2x+5)\,D + 2\}\,y$]

Let
$$(D-2)\ y = v. \qquad \text{...(2)}$$

Then (1) gives $\{(x+3)\,D - 1\}\ v = (x+3)^2 e^x$ or $(x+3)\dfrac{dv}{dx} - v = (x+3)^2\,e^x$

or
$$\frac{dv}{dx} - \frac{1}{x+3}\,v = (x+3)\,e^x, \text{which is linear.}$$

$$\therefore \qquad \text{I. F.} = e^{-\int \frac{dx}{x+3}} = e^{-\log(x+3)} = \frac{1}{x+3}.$$

$$\therefore \qquad v \cdot \frac{1}{x+3} = \int \frac{1}{x+3} \cdot (x+3)\,e^x\,dx + c_1 = e^x + c_1.$$

$$\therefore \qquad v = e^x\,(x+3) + c_1\,(x+3).$$

Putting the value of v in (2), we get

$$(D-2)\ y = e^x\,(x+3) + c_1\,(x+3)$$

or
$$\frac{dy}{dx} - 2\,y = e^x\,(x+3) + c_1\,(x+3), \text{ which is again linear.}$$

$$\therefore \qquad \text{I. F.} = e^{\int -2\,dx} = e^{-2x}.$$

$$\therefore \qquad ye^{-2x} = \int e^{-2x}\,(x+3)\,dx + c_1 \int e^{-2x}\,(x+3)\,dx + c_2$$

$$= -(x+3)\,e^{-x} - e^{-x} - \frac{1}{2}\,c_1\,(x+3)\,e^{-2x} - \frac{1}{4}\,c_1e^{-2x} + c_2$$

or
$$y = -(x+3)\,e^x - e^x - \frac{1}{2}\,c_1\,(x+3) - \frac{1}{4}\,c_1 + c_2e^{2x}$$

or $$y = -xe^x - 4e^x - \frac{1}{4}c_1(2x+7) + c_2e^{2x}$$

or $$y = -xe^x - 4e^x + A(2x+7) + Be^{2x},$$

which is the required general solution of the given differential equation.

Comprehensive Exercise 5

Solve the following equations by the method of factorization of the operator:

1. $x\dfrac{d^2y}{dx^2} + (x-2)\dfrac{dy}{dx} - 2y = x^3.$
2. $x\dfrac{d^2y}{dx^2} + (x-1)\dfrac{dy}{dx} - y = x^2.$ **(Kumaun 2010, 12)**
3. $3x^2D^2y + (2-6x^2)Dy - 4y = 0.$
4. $x^2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - (1+x^2)y = e^{-x}.$
5. $[xD^2 + (1-x)D - 2(1+x)]y = e^{-x}(1-6x).$

Answers 5

1. $y = x^3 + a(x^2 - 2x + 2) + be^{-x}$
2. $y = c_1(x-1) + c_2e^{-x} + x^2$
3. $y = c_1e^{2x}\int e^{(-2x)+2/(3x)}dx + c_2e^{2x}$
4. $y = c_1e^x\int e^{-2x+(1/x)}dx + c_2e^x - \frac{1}{2}e^{-x}$
5. $y = xe^{-x} + c_1e^{2x}\int \frac{1}{x}e^{-3x}\,dx + c_2e^{2x}$

9 Guidelines of the Procedure for the Solution of Linear Differential Equations of Second Order

Before closing this chapter we give the guidelines of the procedure to be applied for the solution of linear differential equation of the second order.

1. First of all put the differential equation in the standard form
$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$
2. Then try to find by inspection an integral belonging to the complementary function of the given differential equation. If it is found, then proceed as in article 2.

3. If a part of the C.F. is not found by inspection, then find the value of $Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2$. If it is a constant or a constant divided by x^2, then the normal form of the equation is easily integrable and so proceed as in article 4.

4. In case the methods given in (2) and (3) do not succeed, the method of change of independent variable given in article 5 may be tried.

In this method we often choose z such that $\frac{Q}{(dz/dx)^2}$ = some suitable constant.

5. In some cases the method of operational factors solves the equation easily.

6. If the students are instructed in the question to solve the equation by the method of variation of parameters, then only the method of variation of parameters must be applied.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. $y = x^m$ is a part of C.F. of equation $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ if

(a) $m(m-1)+Pmx+Qx^2=0$
(b) $m(m+1)+Pmx+Qx^2=0$
(c) $m(m-1)+Pm^2x+Qx^2=0$
(d) none of these

(Kumaun 2008, 13)

2. $y = e^x$ is a solution of $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ if

(a) $1-P+Q=0$
(b) $1+P-Q=0$
(c) $1+P+Q=0$
(d) $1-P-Q=0$

(Kumaun 2007)

3. $y = e^{-x}$ is a part of C.F. of $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ if

(a) $1+P+Q=0$
(b) $1-P+Q=0$
(c) $1-P-Q=0$
(d) $1+P-Q=0$

(Kumaun 2010)

4. $y = e^{mx}$ is a part of C.F. of $\left(\frac{d^2y}{dx^2}\right)+P\left(\frac{dy}{dx}\right)+Qy=R$ if

(a) $m^2+mP+Q=0$
(b) $m(m-1)+Pmx+Qx^2=0$
(c) $m^2+mQ+P=0$
(d) none of these

5. $y = x$ is a solution of $\left(\frac{d^2 y}{dx^2}\right) + P\left(\frac{dy}{dx}\right) + Q\,y = R$ if

(a) $P + Q = 0$ (b) $P - Qx = 0$

(c) $P + Qx = 0$ (d) $P + \frac{Q}{x} = 0$

(Kumaun 2011; Kanpur 16)

6. $y = x^2$ is a part of C.F. of equation $\left(\frac{d^2 y}{dx^2}\right) + P\left(\frac{dy}{dx}\right) + Q\,y = R$ if

(a) $P + Qx = 0$ (b) $1 + P + Q = 0$

(c) $1 - P + Q = 0$ (d) $2 + 2Px + Qx^2 = 0$

(Kumaun 2009)

7. $y = e^{2x}$ is a part of C.F. of the differential equation $\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Q\,y = 0$ if

(a) $2 + 2P + Q = 0$ (b) $4 + 2P + Q = 0$

(c) $4 - 2P + Q = 0$ (d) $4 + 2P - Q = 0$

(Garhwal 2008, 12, 13)

8. A part of the C.F. of the differential equation

$$\frac{d^2 y}{dx^2} - \cot x\frac{dy}{dx} - (1 - \cot x)\,y = e^x \sin x$$

(a) $y = x$ (b) $y = x^2$

(c) $y = e^x$ (d) $y = e^{-x}$ **(Garhwal 2014)**

9. Normal form of second order differential equation

$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Q = R$ is given by $\frac{d^2 V}{dx^2} + IV = S,$ where I and S are

(a) $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4},\ S = Re^{\frac{1}{2}\int P\,dx}$

(b) $I = Q - \frac{1}{3}\frac{dP}{dx} - \frac{P^2}{4},\ S = \frac{R}{2}e^{\frac{1}{2}\int P\,dx}$

(c) $I = Q + \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4},\ S = Re^{\frac{1}{2}\int P\,dx}$

(d) $I = Q - \frac{1}{2}\frac{dP}{dx} + \left(\frac{-P}{4}\right)^2,\ S = Re^{\frac{1}{2}\int P\,dx}$

(Kanpur 2016)

10. If $y_1(x)$ and $y_2(x)$ are solutions of $y'' + 3y' + 2y = e^{e^x}$, then Wronskian $W(y_1\, y_2)$ is

(a) e^{-3x} (b) e^{3x}

(c) $-e^{-3x}$ (d) $2e^{-3x}$

(Kanpur 2016)

11. On solving by changing the independent variable the differential equation $(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + 4y = 0$ reduces into $\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$ where P_1, Q_1, R_1 are

(a) $P_1 = 0, Q_1 = 4, R_1 = 0$ (b) $P_1 = 2, Q_1 = 4, R_1 = 0$

(c) $P_1 = 0, Q_1 = 2, R_1 = 0$ (d) $P_1 = 0, Q_1 = 1, R_1 = 0$

(Kanpur 2016)

12. Normal form of second order differential equation $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = (\sec x)e^x$ is given by $\frac{d^2v}{dx^2} + Xv = Y$, where X and Y are

(a) $X = 6, Y = e^x$ (b) $X = -6, Y = e^x$

(c) $X = 6, Y = e^{-x}$ (d) $X = -6, Y = e^{-x}$

13. Normal form of second order differential equation $\frac{d}{dx}(\cos^2 x \frac{dy}{dx}) + y\cos^2 x = 0$ is given by $\frac{d^2v}{dx^2} + Xv = Y$, where X and Y are

(a) $X = -2, Y = 1$ (b) $X = -2, Y = -1$

(c) $X = 2, Y = 0$ (d) $X = 0, Y = 2$

14. On solving by changing the independent variable the differential equation $\frac{d^2y}{dx^2} - (1+4e^x)\frac{dy}{dx} + 3e^{2x}y = e^{2(x+e^x)}$, reduces into $\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$ where P_1, Q_1, R_1 are

(a) $P_1 = -4, Q_1 = 3, R_1 = e^{2z}$ (b) $P_1 = 4, Q_1 = -3, R_1 = e^{2z}$

(c) $P_1 = -4, Q_1 = 3, R_1 = e^{-2z}$ (d) $P_1 = -4, Q_1 = -3, R_1 = e^{2z}$

15. If u and v are solutions of $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$, then Wronskian $W(u\ v)$ is

(a) -2 (b) 2

(c) 0 (d) 1

Fill in The Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. $y = x^m$ is a part of C.F. of equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ if ......

(Meerut 2003)

2. $y = e^x$ is a solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ if ......

3. $y = e^{-x}$ is a part of C.F. of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ if ......

4. $y = e^{mx}$ is a part of C.F. of $\left(\frac{d^2y}{dx^2}\right) + P\left(\frac{dy}{dx}\right) + Qy = R$ if ......

5. $y = x$ is a solution of $\left(\frac{d^2 y}{dx^2}\right) + P\left(\frac{dy}{dx}\right) + Qy = R$ if ......

6. $y = x^2$ is a part of C.F. of equation $\left(\frac{d^2 y}{dx^2}\right) + P\left(\frac{dy}{dx}\right) + Qy = R$ if ......

True or False

Write 'T' for true and 'F' for false statement.

1. $y = x^2$ is a part of C.F. of the solution of equation

$$(1-x)\frac{d^2 y}{dx^2} + x\frac{dy}{dx} - y = (1-x)^2.$$

2. $y = e^x$ is a part of C.F. of the solution of equation

$$x\frac{d^2 y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)\,y = 0\,.$$

3. $y = x$ is a part of C.F. of the solution of equation

$$x\frac{dy}{dx} - y = (x-1)\left(\frac{d^2 y}{dx^2} - x + 1\right).$$

4. $\frac{d^2 y}{dx^2} + \left(1 + \frac{2}{x}\cot x - \frac{2}{x^2}\right)y = x\cos x$ is a linear differential equation of second order.

Answers

Multiple Choice Questions

1. (a) 2. (c) 3. (b) 4. (a) 5. (c)
6. (d) 7. (b) 8. (d) 9. (a) 10. (c)
11. (d) 12. (a) 13. (c) 14. (a) 15. (b)

Fill in the Blank(s)

1. $m(m-1) + Pmx + Qx^2 = 0$
2. $1 + P + Q = 0$
3. $1 - P + Q = 0$
4. $m^2 + Pm + Q = 0$
5. $P + Qx = 0$
6. $2 + 2Px + Qx^2 = 0$

True or False

1. *F* 2. *T* 3. *T* 4. *T*

□

Chapter

8

Differentiation of Vectors

1 Vector Function

We know that a scalar quantity possesses only magnitude and has no concern with direction. A single real number gives us a complete representation of a scalar quantity. Thus a scalar quantity is nothing but a real number.

Let D be any subset of the set of all real numbers. If to each element t of D, we associate by some rule a unique real number $f(t)$, then this rule defines a **scalar function** of the scalar variable t. Here $f(t)$ is a scalar quantity and thus f is a scalar function.

In a similar manner we define a vector function.

Let D be any subset of the set of all real numbers. If to each element t of D, we associate by some rule a unique vector $\mathbf{f}(t)$, *then this rule defines a* **vector function** *of the scalar variable t. Here* $\mathbf{f}(t)$ *is a vector quantity and thus* $\mathbf{f}$ *is a vector function.*

We know that every vector can be uniquely expressed as a linear combination of three fixed non-coplanar vectors. Therefore we may write

$$\mathbf{f}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$$

where **i** , **j** , **k** denote a fixed right handed triad of three mutually perpendicular non-coplanar unit vectors.

2 Scalar Fields and Vector Fields

If to each point $P(x\ ,\ y\ ,\ z)$ of a region R in space there corresponds a unique scalar $f(P)$, then f is called a **scalar point function** and we say that a **scalar field** f has been defined in R .

Examples: (1) The temperature at any point within or on the surface of earth at a certain time defines a scalar field.
(2) $f(x\ ,\ y\ ,\ z) = x^2 - y^3 - 3z^2$ defines a scalar field.

If to each point $P\ (x\ ,\ y\ ,\ z)$ of a region R in space there corresponds a unique vector $\mathbf{f}(P)$, then **f** is called a **vector point function** and we say that a **vector field f** has been defined in R.

Examples: (1) If the velocity at any point $(x,\ y,\ z)$ of a particle moving in a curve is known at a certain time, then a vector field is defined.
(2) $\mathbf{f}(x\ ,\ y\ ,\ z) = xy^2\,\mathbf{i} + 3yz^3\,\mathbf{j} - 2x^2z\mathbf{k}$ defines a vector field.

3 Limit and Continuity of a Vector Function

Definition 1: *A vector function* $\mathbf{f}(t)$ *is said to tend to a limit* $\mathbf{l}$, *when* t *tends to* t_0 , *if for any given positive number* ε , *however small, there corresponds a positive number* δ *such that*

$$|\mathbf{f}(t) - \mathbf{l}\ | < \varepsilon \quad \textit{whenever} \quad 0 < |t - t_0\ | < \delta.$$

If $\mathbf{f}(t)$ tends to a limit $\mathbf{l}$ as t tends to t_0 , we write $\lim\limits_{t \to t_0} \mathbf{f}(t) = \mathbf{l}$.

Definition 2: *A vector function* $\mathbf{f}(t)$ *is said to be continuous for a value* t_0 *of* t *if*

(i) $\mathbf{f}(t_0)$ *is defined and*

(ii) for any given positive number ε, *however small, there corresponds a positive number* δ *such that* $|\mathbf{f}(t) - \mathbf{f}(t_0)| < \varepsilon$, *whenever* $|t - t_0| < \delta$.

Further a vector function $\mathbf{f}\ (t)$ *is said to be continuous if it is continuous for every value of* t *for which it has been defined.*

We shall give here (without proof) some important results about the limits and continuity of a vector function.

Theorem 1: *The necessary and sufficient condition for a vector function* $\mathbf{f}(t)$ *to be* continuous at $t = t_0$ is that $\lim\limits_{t \to t_0} \mathbf{f}(t) = \mathbf{f}(t_0)$.

(Purvanchal 2014)

Theorem 2: *If* $\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ *, then* $\mathbf{f}(t)$ *is continuous if and only if* $f_1(t),\ f_2(t),\ f_3(t)$ *are continuous.*

Theorem 3: *Let* $\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ *and* $\mathbf{l} = l_1\mathbf{i} + l_2\mathbf{j} + l_3\mathbf{k}$.

Then the necessary and sufficient conditions that $\lim\limits_{t \to t_0} \mathbf{f}(t) = \mathbf{l}$ *are*

$$\lim_{t \to t_0} f_1(t) = l_1\ ,\ \lim_{t \to t_0} f_2(t) = l_2 \text{ and } \lim_{t \to t_0} f_3(t) = l_3\ .$$

Theorem 4: *If* $\mathbf{f}(t), \mathbf{g}(t)$ *are vector functions of scalar variable* t *and* $\phi(t)$ *is a scalar function of scalar variable* t *, then*

(i) $\lim\limits_{t \to t_0} [\mathbf{f}(t) \pm \mathbf{g}(t)] = \lim\limits_{t \to t_0} \mathbf{f}(t) \pm \lim\limits_{t \to t_0} \mathbf{g}(t)$

(ii) $\lim\limits_{t \to t_0} [\mathbf{f}(t) \bullet \mathbf{g}(t)] = \left[\lim\limits_{t \to t_0} \mathbf{f}(t)\right] \bullet \left[\lim\limits_{t \to t_0} \mathbf{g}(t)\right]$

(iii) $\lim\limits_{t \to t_0} [\mathbf{f}(t) \times \mathbf{g}(t)] = \left[\lim\limits_{t \to t_0} \mathbf{f}(t)\right] \times \left[\lim\limits_{t \to t_0} \mathbf{g}(t)\right]$

(iv) $\lim\limits_{t \to t_0} [\phi(t)\, \mathbf{f}(t)] = \left[\lim\limits_{t \to t_0} \phi(t)\right]\left[\lim\limits_{t \to t_0} \mathbf{f}(t)\right]$

(v) $\lim\limits_{t \to t_0} |\mathbf{f}(t)| = \left|\lim\limits_{t \to t_0} \mathbf{f}(t)\right|$.

4 Derivative of a Vector Function with Respect to a Scalar

Definition: *Let* $\mathbf{r} = \mathbf{f}(t)$ *be a vector function of the scalar variable t. We define*

$$\mathbf{r} + \delta\mathbf{r} = \mathbf{f}(t + \delta t).$$

$$\therefore \quad \delta\mathbf{r} = \mathbf{f}(t + \delta t) - \mathbf{f}(t).$$

Consider the vector $\dfrac{\delta\mathbf{r}}{\delta t} = \dfrac{\mathbf{f}(t + \delta t) - \mathbf{f}(t)}{\delta t}$.

If $\lim\limits_{\delta t \to 0} \dfrac{\delta\mathbf{r}}{\delta t} = \lim\limits_{\delta t \to 0} \dfrac{\mathbf{f}(t + \delta t) - \mathbf{f}(t)}{\delta t}$ *exists, then the value of this limit, which we shall denote by* $\dfrac{d\mathbf{r}}{dt}$ *, is called the derivative of the vector function* $\mathbf{r}$ *with respect to the scalar t.*

Symbolically

$$\frac{d\mathbf{r}}{dt} = \lim_{\delta t \to 0} \frac{(\mathbf{r} + \delta\mathbf{r}) - \mathbf{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\mathbf{f}(t + \delta t) - \mathbf{f}(t)}{\delta t}\ .$$

If $\dfrac{d\mathbf{r}}{dt}$ exists, then $\mathbf{r}$ is said to be differentiable. Since $\dfrac{\delta\mathbf{r}}{\delta t}$ is a vector quantity, therefore $\dfrac{d\mathbf{r}}{dt}$ is also a vector quantity.

Successive Derivatives: If we differentiate $\frac{d\mathbf{r}}{dt}$ again, we get $\frac{d^2\mathbf{r}}{dt^2}$ which is called the second derivative of $\mathbf{r}$ w.r.t. t, and so on.

Thus, we continue differentiating successively upto n times and get

$$\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3}, \ldots\ldots, \frac{d^n\mathbf{r}}{dt^n},$$

where $\frac{d^n\mathbf{r}}{dt^n}$ is called the n^{th} diff. coeff. (or derivative) of $\mathbf{r}$ w.r.t. t. (Here diff. coeff. means differential coefficient.)

We often represent $\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \ldots\ldots$ by $\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \ldots\ldots$ respectively.

A scalar or vector function of t is called differentiable of order n if its n^{th} order derivative exists.

5 Differentiation Formulae

Theorem: *If* **a**, **b** *and* **c** *are differentiable vector functions of a scalar t and* ϕ *is a differentiable scalar function of the same variable t, then*

1. $\frac{d}{dt}(\mathbf{a}+\mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}$
2. $\frac{d}{dt}(\mathbf{a} \bullet \mathbf{b}) = \mathbf{a} \bullet \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \bullet \mathbf{b}$
3. $\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \times \mathbf{b}$
4. $\frac{d}{dt}(\phi\mathbf{a}) = \phi\frac{d\mathbf{a}}{dt} + \frac{d\phi}{dt}\mathbf{a}$
5. $\frac{d}{dt}[\mathbf{a}\ \mathbf{b}\,\mathbf{c}] = \left[\frac{d\mathbf{a}}{dt}\ \mathbf{b}\,\mathbf{c}\right] + \left[\mathbf{a}\ \frac{d\mathbf{b}}{dt}\ \mathbf{c}\right] + \left[\mathbf{a}\ \mathbf{b}\frac{d\mathbf{c}}{dt}\right]$ **(Meerut 2013B; Purvanchal 08)**
6. $\frac{d}{dt}\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\} = \frac{d\mathbf{a}}{dt} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \times \left(\frac{d\mathbf{b}}{dt} \times \mathbf{c}\right) + \mathbf{a} \times \left(\mathbf{b} \times \frac{d\mathbf{c}}{dt}\right)$.

(Meerut 2000; Purvanchal 08)

Proof: 1. $\frac{d}{dt}(\mathbf{a}+\mathbf{b}) = \lim_{\delta t \to 0} \frac{\{(\mathbf{a}+\delta\mathbf{a}) + (\mathbf{b}+\delta\mathbf{b})\} - (\mathbf{a}+\mathbf{b})}{\delta t}$

$$= \lim_{\delta t \to 0} \frac{\delta\mathbf{a}+\delta\mathbf{b}}{\delta t} = \lim_{\delta t \to 0}\left(\frac{\delta\mathbf{a}}{\delta t} + \frac{\delta\mathbf{b}}{dt}\right)$$

$$= \lim_{\delta t \to 0}\frac{\delta\mathbf{a}}{\delta t} + \lim_{\delta t \to 0}\frac{\delta\mathbf{b}}{dt} = \frac{d\,\mathbf{a}}{dt} + \frac{d\,\mathbf{b}}{dt}.$$

Thus the derivative of the sum of two vectors is equal to the sum of their derivatives, as it is also in Scalar Calculus.

Similarly we can prove that $\frac{d}{dt}(\mathbf{a}-\mathbf{b})=\frac{d\mathbf{a}}{dt}-\frac{d\mathbf{b}}{dt}\cdot$

In general if $\mathbf{r}_1$, $\mathbf{r}_2$, ……, $\mathbf{r}_n$ are vector functions of a scalar t, then

$$\frac{d}{dt}(\mathbf{r}_1+\mathbf{r}_2+\ldots\ldots+\mathbf{r}_n)=\frac{d\mathbf{r}_1}{dt}+\frac{d\mathbf{r}_2}{dt}+\ldots\ldots+\frac{d\mathbf{r}_n}{dt}\cdot$$

2. $$\frac{d}{dt}(\mathbf{a}\bullet\mathbf{b})=\lim_{\delta t\to 0}\frac{(\mathbf{a}+\delta\mathbf{a})\bullet(\mathbf{b}+\delta\mathbf{b})-\mathbf{a}\bullet\mathbf{b}}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\mathbf{a}\bullet\mathbf{b}+\mathbf{a}\bullet\delta\mathbf{b}+\delta\mathbf{a}\bullet\mathbf{b}+\delta\mathbf{a}\bullet\delta\mathbf{b}-\mathbf{a}\bullet\mathbf{b}}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\mathbf{a}\bullet\delta\mathbf{b}+\delta\mathbf{a}\bullet\mathbf{b}+\delta\mathbf{a}\bullet\delta\mathbf{b}}{\delta t}$$

$$=\lim_{\delta t\to 0}\left\{\mathbf{a}\bullet\frac{\delta\mathbf{b}}{\delta t}+\frac{\delta\mathbf{a}}{\delta t}\bullet\mathbf{b}+\frac{\delta\mathbf{a}}{\delta t}\bullet\delta\mathbf{b}\right\}$$

$$=\lim_{\delta t\to 0}\mathbf{a}\bullet\frac{\delta\mathbf{b}}{\delta t}+\lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t}\bullet\mathbf{b}+\lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t}\bullet\delta\mathbf{b}$$

$$=\mathbf{a}\bullet\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\bullet\mathbf{b}+\frac{d\mathbf{a}}{dt}\bullet\mathbf{0},\qquad \text{since }\delta\mathbf{b}\to\text{ zero vector as }\delta t\to 0$$

$$=\mathbf{a}\bullet\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\bullet\mathbf{b}$$

Note: We know that $\mathbf{a}\bullet\mathbf{b}=\mathbf{b}\bullet\mathbf{a}$. Therefore while evaluating $\frac{d}{dt}(\mathbf{a}\bullet\mathbf{b})$, we should not bother about the order of the factors.

3. $$\frac{d}{dt}(\mathbf{a}\times\mathbf{b})=\lim_{\delta t\to 0}\frac{(\mathbf{a}+\delta\mathbf{a})\times(\mathbf{b}+\delta\mathbf{b})-\mathbf{a}\times\mathbf{b}}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\delta\mathbf{b}+\delta\mathbf{a}\times\mathbf{b}+\delta\mathbf{a}\times\delta\mathbf{b}-\mathbf{a}\times\mathbf{b}}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\mathbf{a}\times\delta\mathbf{b}+\delta\mathbf{a}\times\mathbf{b}+\delta\mathbf{a}\times\delta\mathbf{b}}{\delta t}$$

$$=\lim_{\delta t\to 0}\left\{\mathbf{a}\times\frac{\delta\mathbf{b}}{\delta t}+\frac{\delta\mathbf{a}}{\delta t}\times\mathbf{b}+\frac{\delta\mathbf{a}}{\delta t}\times\delta\mathbf{b}\right\}$$

$$=\lim_{\delta t\to 0}\mathbf{a}\times\frac{\delta\mathbf{b}}{\delta t}+\lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t}\times\mathbf{b}+\lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t}\times\delta\mathbf{b}$$

$$=\mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b}+\frac{d\mathbf{a}}{dt}\times\mathbf{0},\ \text{since }\delta\mathbf{b}\to\text{ zero vector as }\delta t\to 0$$

$$=\mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b}+\mathbf{0}=\mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b}.$$

Note: We know that cross product of two vectors is not commutative because $\mathbf{a}\times\mathbf{b}=-\,\mathbf{b}\times\mathbf{a}$. Therefore while evaluating $\frac{d}{dt}(\mathbf{a}\times\mathbf{b})$, we must maintain the order of the factors $\mathbf{a}$ and $\mathbf{b}$.

4. $$\frac{d}{dt}(\phi\mathbf{a}) = \lim_{\delta t\to 0}\frac{(\phi+\delta\phi)(\mathbf{a}+\delta\mathbf{a})-\phi\mathbf{a}}{\delta t}$$

$$= \lim_{\delta t\to 0}\frac{\phi\mathbf{a}+\phi\delta\mathbf{a}+\delta\phi\mathbf{a}+\delta\phi\delta\mathbf{a}-\phi\mathbf{a}}{\delta t}$$

$$= \lim_{\delta t\to 0}\frac{\phi\delta\mathbf{a}+\delta\phi\mathbf{a}+\delta\phi\delta\mathbf{a}}{\delta t}$$

$$= \lim_{\delta t\to 0}\left\{\phi\frac{\delta\mathbf{a}}{\delta t}+\frac{\delta\phi}{\delta t}\mathbf{a}+\frac{\delta\phi}{\delta t}\delta\mathbf{a}\right\}$$

$$= \lim_{\delta t\to 0}\phi\frac{\delta\mathbf{a}}{\delta t}+\lim_{\delta t\to 0}\frac{\delta\phi}{\delta t}\mathbf{a}+\lim_{\delta t\to 0}\frac{\delta\phi}{\delta t}\delta\mathbf{a}$$

$$= \phi\frac{d\mathbf{a}}{dt}+\frac{d\phi}{dt}\mathbf{a}+\frac{d\phi}{dt}\mathbf{0}, \text{ since } \delta\mathbf{a}\to\text{zero vector as } \delta t\to 0$$

$$= \phi\frac{d\mathbf{a}}{dt}+\frac{d\phi}{dt}\mathbf{a}+\mathbf{0} = \phi\frac{d\mathbf{a}}{dt}+\frac{d\phi}{dt}\mathbf{a}.$$

Note: $\phi\mathbf{a}$ is the multiplication of a vector by a scalar. In the case of such multiplication we usually write the scalar in the first position and the vector in the second position.

5. $$\frac{d}{dt}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \frac{d}{dt}\{\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})\} = \mathbf{a}\bullet\frac{d}{dt}(\mathbf{b}\times\mathbf{c})+\frac{d\mathbf{a}}{dt}\bullet(\mathbf{b}\times\mathbf{c}) \qquad \text{[by rule (2)]}$$

$$= \mathbf{a}\bullet\left(\mathbf{b}\times\frac{d\mathbf{c}}{dt}+\frac{d\mathbf{b}}{dt}\times\mathbf{c}\right)+\frac{d\mathbf{a}}{dt}\bullet(\mathbf{b}\times\mathbf{c}) \qquad \text{[by rule (3)]}$$

$$= \mathbf{a}\bullet\left(\mathbf{b}\times\frac{d\mathbf{c}}{dt}\right)+\mathbf{a}\bullet\left(\frac{d\mathbf{b}}{dt}\times\mathbf{c}\right)+\frac{d\mathbf{a}}{dt}\bullet(\mathbf{b}\times\mathbf{c})$$

$$= \left[\mathbf{a}\ \mathbf{b}\ \frac{d\mathbf{c}}{dt}\right]+\left[\mathbf{a}\ \frac{d\mathbf{b}}{dt}\ \mathbf{c}\right]+\left[\frac{d\mathbf{a}}{dt}\ \mathbf{b}\ \mathbf{c}\right]$$

$$= \left[\frac{d\mathbf{a}}{dt}\ \mathbf{b}\ \mathbf{c}\right]+\left[\mathbf{a}\ \frac{d\mathbf{b}}{dt}\ \mathbf{c}\right]+\left[\mathbf{a}\ \mathbf{b}\ \frac{d\mathbf{c}}{dt}\right].$$

Note: Here **[abc]** is the scalar triple product of three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Therefore while evaluating $\frac{d}{dt}$ **[abc]** we must maintain the cyclic order of each factor.

6. $$\frac{d}{dt}\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\} = \mathbf{a}\times\frac{d}{dt}(\mathbf{b}\times\mathbf{c})+\frac{d\mathbf{a}}{dt}\times(\mathbf{b}\times\mathbf{c}) \qquad \text{[by rule (3)]}$$

$$= \mathbf{a}\times\left(\frac{d\mathbf{b}}{dt}\times\mathbf{c}+\mathbf{b}\times\frac{d\mathbf{c}}{dt}\right)+\frac{d\mathbf{a}}{dt}\times(\mathbf{b}\times\mathbf{c})$$

$$= \mathbf{a} \times \left(\frac{d\mathbf{b}}{dt} \times \mathbf{c} \right) + \mathbf{a} \times \left(\mathbf{b} \times \frac{d\mathbf{c}}{dt} \right) + \frac{d\mathbf{a}}{dt} \times (\mathbf{b} \times \mathbf{c})$$

$$= \frac{d\mathbf{a}}{dt} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \times \left(\frac{d\mathbf{b}}{dt} \times \mathbf{c} \right) + \mathbf{a} \times \left(\mathbf{b} \times \frac{d\mathbf{c}}{dt} \right).$$

6 Derivative of a Function of a Function

Suppose **r** is a differentiable vector function of a scalar variable s and s is a differentiable scalar function of another scalar variable t. Then **r** is a function of t.

An increment δt in t produces an increment $\delta \mathbf{r}$ in **r** and an increment δs in s. When $\delta t \to 0$, $\delta \mathbf{r} \to \mathbf{0}$ and $\delta s \to 0$.

We have $$\frac{d\mathbf{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \lim_{\delta t \to 0} \left(\frac{\delta s}{\delta t} \frac{\delta \mathbf{r}}{\delta s} \right)$$

$$= \left(\lim_{\delta t \to 0} \frac{\delta s}{\delta t} \right) \left(\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta s} \right) = \frac{ds}{dt} \frac{d\mathbf{r}}{ds}.$$

7 Derivative of a Constant Vector

A vector is said to be constant only if both its magnitude and direction are fixed. If either of these changes then the vector will change and thus it will not be constant.

Let **r** be a constant vector function of the scalar variable t. Let $\mathbf{r} = \mathbf{c}$, where **c** is a constant vector. Then $\mathbf{r} + \delta \mathbf{r} = \mathbf{c}$.

$\therefore$ $\delta \mathbf{r} = \mathbf{0}$ (zero vector). $\therefore$ $\dfrac{\delta \mathbf{r}}{\delta t} = \dfrac{\mathbf{0}}{\delta t} = \mathbf{0}.$

$\therefore$ $\lim\limits_{\delta t \to 0} \dfrac{\delta \mathbf{r}}{\delta t} = \lim\limits_{\delta t \to 0} \mathbf{0} = \mathbf{0}.$ $\therefore$ $\dfrac{d\mathbf{r}}{dt} = \mathbf{0}$ (zero vector).

Thus the derivative of a constant vector is equal to the null vector.

8 Derivative of a Vector Function in Terms of its Components

Let **r** be a vector function of the scalar variable t. **(Avadh 2014)**

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where the components x, y, z are scalar functions of the scalar variable t and **i, j, k** are fixed unit vectors.

We have $\mathbf{r} + \delta \mathbf{r} = (x + \delta x)\,\mathbf{i} + (y + \delta y)\,\mathbf{j} + (z + \delta z)\,\mathbf{k}.$

$\therefore$ $\delta \mathbf{r} = (\mathbf{r} + \delta \mathbf{r}) - \mathbf{r} = \delta x\,\mathbf{i} + \delta y\mathbf{j} + \delta z\mathbf{k}.$

$\therefore$ $$\frac{\delta \mathbf{r}}{\delta t} = \frac{\delta x}{\delta t}\mathbf{i} + \frac{\delta y}{\delta t}\mathbf{j} + \frac{\delta z}{\delta t}\mathbf{k}.$$

$\therefore$ $$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \lim_{\delta t \to 0} \left\{ \frac{\delta x}{\delta t}\mathbf{i} + \frac{\delta y}{\delta t}\mathbf{j} + \frac{\delta z}{\delta t}\mathbf{k} \right\}.$$

$$\therefore \qquad \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}.$$

Thus in order to differentiate a vector we should differentiate its components.

Note: If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then sometimes we also write it as $\mathbf{r} = (x, y, z)$. In this notation $\dfrac{d\mathbf{r}}{dt} = \left(\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}\right), \dfrac{d^2\mathbf{r}}{dt^2} = \left(\dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2}, \dfrac{d^2z}{dt^2}\right)$, and so on.

Alternative Method:

We have $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are constant vectors and so their derivatives will be zero.

Now,
$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{d}{dt}(x\mathbf{i}) + \frac{d}{dt}(y\mathbf{j}) + \frac{d}{dt}(z\mathbf{k})$$
$$= \frac{dx}{dt}\,\mathbf{i} + x\,\frac{d\mathbf{i}}{dt} + \frac{dy}{dt}\,\mathbf{j} + y\,\frac{d\mathbf{j}}{dt} + \frac{dz}{dt}\,\mathbf{k} + z\,\frac{d\mathbf{k}}{dt}$$
$$= \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}\text{, since } \frac{d\mathbf{i}}{dt}\text{, etc. vanish.}$$

9 Some Important Results

Theorem 1: *The necessary and sufficient condition for the vector function* $\mathbf{a}(t)$ *to be constant is that* $\dfrac{d\mathbf{a}}{dt} = \mathbf{0}.$

(Purvanchal 2014)

Proof: The condition is necessary. Let $\mathbf{a}(t)$ be a constant vector function of the scalar variable t. Then $\mathbf{a}(t + \delta t) = \mathbf{a}(t)$. We have

$$\frac{d\mathbf{a}}{dt} = \lim_{\delta t \to 0} \frac{\mathbf{a}(t + \delta t) - \mathbf{a}(t)}{\delta t} = \lim_{\delta t \to 0} \frac{\mathbf{0}}{\delta t} = \mathbf{0}.$$

Therefore the condition is necessary.

The condition is sufficient. Let $\dfrac{d\mathbf{a}}{dt} = \mathbf{0}.$ Then to prove that $\mathbf{a}$ is a constant vector. Let

$$\mathbf{a}(t) = a_1(t)\,\mathbf{i} + a_2(t)\,\mathbf{j} + a_3(t)\,\mathbf{k}.$$

Then
$$\frac{d\mathbf{a}}{dt} = \frac{da_1}{dt}\,\mathbf{i} + \frac{da_2}{dt}\,\mathbf{j} + \frac{da_3}{dt}\,\mathbf{k}.$$

Therefore $\dfrac{d\mathbf{a}}{dt} = \mathbf{0}$ gives, $\dfrac{da_1}{dt}\,\mathbf{i} + \dfrac{da_2}{dt}\,\mathbf{j} + \dfrac{da_3}{dt}\,\mathbf{k} = \mathbf{0}.$

Equating to zero the coefficients of **i, j** and **k**, we get

$$\frac{da_1}{dt} = 0, \frac{da_2}{dt} = 0, \frac{da_3}{dt} = 0.$$

Hence a_1, a_2, a_3 are constant scalars *i.e.*, they are independent of t. Therefore $\mathbf{a}(t)$ is a constant vector function.

Theorem 2: *If* **a** *is a differentiable vector function of the scalar variable t and if* $|\mathbf{a}| = a$, *then*

(i) $\dfrac{d}{dt}(\mathbf{a}^2) = 2a\dfrac{da}{dt}$; (ii) $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = a\dfrac{da}{dt}\cdot$

Proof: (i) We have $\mathbf{a}^2 = \mathbf{a} \bullet \mathbf{a} = (a)(a)\cos 0 = a^2$.

Therefore $\dfrac{d}{dt}(\mathbf{a}^2) = \dfrac{d}{dt}(a^2) = 2a\dfrac{da}{dt}\cdot$

(ii) We have $\dfrac{d}{dt}(\mathbf{a}^2) = \dfrac{d}{dt}(\mathbf{a} \bullet \mathbf{a}) = \dfrac{d\mathbf{a}}{dt} \bullet \mathbf{a} + \mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 2\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt}\cdot$

Also $\dfrac{d}{dt}(\mathbf{a}^2) = \dfrac{d}{dt}(a^2) = 2a\dfrac{da}{dt}\cdot$

$\therefore$ $2\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 2a\dfrac{da}{dt}$ or $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = a\dfrac{da}{dt}\cdot$

Theorem 3: *If* **a** *has constant length (fixed magnitude), then* **a** *and* $\dfrac{d\mathbf{a}}{dt}$ *are perpendicular provided* $\left|\dfrac{d\mathbf{a}}{dt}\right| \neq 0$.

Proof: Let $|\mathbf{a}| = a =$ constant. Then $\mathbf{a} \bullet \mathbf{a} = a^2 =$ constant.

$\therefore$ $\dfrac{d}{dt}(\mathbf{a} \bullet \mathbf{a}) = 0$ or $\dfrac{d\mathbf{a}}{dt} \bullet \mathbf{a} + \mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0$

or $2\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0$ or $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0$.

Thus the scalar product of two vectors **a** and $\dfrac{d\mathbf{a}}{dt}$ is zero.

Therefore **a** is perpendicular to $\dfrac{d\mathbf{a}}{dt}$ provided $\dfrac{d\mathbf{a}}{dt}$ is not null vector *i.e.*, provided

$$\left|\frac{d\mathbf{a}}{dt}\right| \neq 0.$$

Thus the derivative of a vector of constant length is perpendicular to the vector provided the vector itself is not constant.

Theorem 4: *The necessary and sufficient condition for the vector* **a** (t) *to have constant magnitude is* $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0$. **(Meerut 2001, 04B, 06, 07, 11, 13; Avadh 13; Kashi 10, 12; Purvanchal 11)**

Proof: Let **a** be a vector function of the scalar variable t. Let $|\mathbf{a}| = a =$ constant. Then $\mathbf{a} \bullet \mathbf{a} = a^2 =$ constant.

$\therefore$ $\dfrac{d}{dt}(\mathbf{a} \bullet \mathbf{a}) = 0$ or $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} + \dfrac{d\mathbf{a}}{dt} \bullet \mathbf{a} = 0$

or $2\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0$ or $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0$.

Therefore the condition is necessary.

Condition is sufficient. If $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0$, then

$$\mathbf{a} \bullet \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{a}}{dt} \bullet \mathbf{a} = 0 \quad \text{or} \quad \frac{d}{dt}(\mathbf{a} \bullet \mathbf{a}) = 0$$

or $\mathbf{a} \bullet \mathbf{a} = \text{constant}$ or $\mathbf{a}^2 = \text{constant}$

or $a^2 = \text{constant}$ or $|\mathbf{a}| = \text{constant}$

Theorem 5: *If* **a** *is a differentiable vector function of the scalar variable t, then*

$$\frac{d}{dt}\left(\mathbf{a} \times \frac{d\mathbf{a}}{dt}\right) = \mathbf{a} \times \frac{d^2\mathbf{a}}{dt^2} \cdot$$

Proof: We have

$$\frac{d}{dt}\left(\mathbf{a} \times \frac{d\mathbf{a}}{dt}\right) = \frac{d\mathbf{a}}{dt} \times \frac{d\mathbf{a}}{dt} + \mathbf{a} \times \frac{d^2\mathbf{a}}{dt^2} = \mathbf{0} + \mathbf{a} \times \frac{d^2\mathbf{a}}{dt^2},$$

since the cross product of two equal vectors $\dfrac{d\mathbf{a}}{dt}$ is zero $= \mathbf{a} \times \dfrac{d^2\mathbf{a}}{dt^2} \cdot$

Theorem 6: *The necessary and sufficient condition for the vector* **a** (*t*) *to have constant direction is* $\mathbf{a} \times \dfrac{d\mathbf{a}}{dt} = \mathbf{0}$.

(Meerut 2004B, 06, 07, 10, 11, 13B; Avadh 14)

Proof: Let **a** be a vector function of the scalar variable t. Let **A** be a unit vector in the direction of **a**. If a be the magnitude of **a**, then $\mathbf{a} = a\mathbf{A}$.

$$\therefore \quad \frac{d\mathbf{a}}{dt} = a\frac{d\mathbf{A}}{dt} + \frac{da}{dt}\mathbf{A}.$$

Hence

$$\begin{aligned}\mathbf{a} \times \frac{d\mathbf{a}}{dt} &= (a\,\mathbf{A}) \times \left(a\frac{d\mathbf{A}}{dt} + \frac{da}{dt}\mathbf{A}\right)\\ &= a^2\mathbf{A} \times \frac{d\mathbf{A}}{dt} + a\frac{da}{dt}\mathbf{A} \times \mathbf{A}\\ &= a^2\mathbf{A} \times \frac{d\mathbf{A}}{dt} \qquad [\because\ \mathbf{A} \times \mathbf{A} = \mathbf{0}] \qquad \ldots(1)\end{aligned}$$

The condition is necessary. Suppose **a** has a constant direction. Then **A** is a constant vector because it has constant direction as well as constant magnitude. Therefore $\dfrac{d\mathbf{A}}{dt} = \mathbf{0}$.

$\therefore$ From (1), we get $\mathbf{a} \times \dfrac{d\mathbf{a}}{dt} = a^2\mathbf{A} \times \mathbf{0} = \mathbf{0}$.

Therefore the condition is necessary.

The condition is sufficient. Suppose that $\mathbf{a} \times \dfrac{d\mathbf{a}}{dt} = \mathbf{0}$.

Then from (1), we get

$$a^2\mathbf{A} \times \frac{d\mathbf{A}}{dt} = \mathbf{0} \quad \text{or} \quad \mathbf{A} \times \frac{d\,\mathbf{A}}{dt} = \mathbf{0}. \qquad \ldots(2)$$

Since **A** is of constant length, therefore $\mathbf{A} \bullet \frac{d\mathbf{A}}{dt} = 0.$ ...(3)

From (2) and (3), we get $\frac{d\mathbf{A}}{dt} = \mathbf{0}$.

Hence **A** is a constant vector *i.e.*, the direction of **a** is constant.

10 Curves in Space

A curve in a three dimensional Euclidean space may be regarded as the intersection of two surfaces represented by two equations of the form

$$F_1 \ (x, y, z) = 0, F_2 \ (x, y, z) = 0.$$

It can be easily seen that the parametric equations of the form

$$x = f_1 (t), y = f_2 (t), z = f_3 (t)$$

where x , y , z are scalar functions of the scalar t, also represent a curve in three dimensional space. Here (x, y, z) are the coordinates of a current point on the curve. The scalar variable t may range over a set of values $a \le t \le b$.

In vector notation an equation of the form $\mathbf{r} = \mathbf{f} \ (t)$, represents a curve in three dimensional space if **r** is the position vector of a current point on the curve. As t changes, **r** will give position vectors of different points on the curve. The vector $\mathbf{f} \ (t)$ can be expressed as

$$f_1 (t)\mathbf{i} + f_2 (t)\mathbf{j} + f_3 (t)\mathbf{k}.$$

Also if (x, y, z) are the coordinates of a current point on the curve whose position vector is **r**, then $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Therefore the single vector equation $\mathbf{r} = \mathbf{f} \ (t)$

i.e., $$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = f_1 (t)\mathbf{i} + f_2 (t)\mathbf{j} + f_3 (t)\mathbf{k}$$

is equivalent to the three parametric equations

$$x = f_1 (t), y = f_2 (t), z = f_3 (t).$$

Thus *a curve in a space may be defined as the locus of a point whose coordinates may be expressed as a function of a* **single** *parameter.*

For example, the two equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \ \ x = a \cosh \frac{z}{a}$$

specify a curve in three dimensional space. The parametric equations of this curve are $$x = a \cosh u, \ \ y = b \sinh u, \ \ z = au.$$

And its vectorial equation is

$$\mathbf{r} = a \cosh u \ \mathbf{i} + b \sinh u \ \mathbf{j} + au \ \mathbf{k}.$$

The vector equation $\mathbf{r} = a \cos t\mathbf{i} + b \sin t\mathbf{j} + 0\mathbf{k}$ represents an ellipse, as for different values of t, the end point of **r** describes an ellipse.

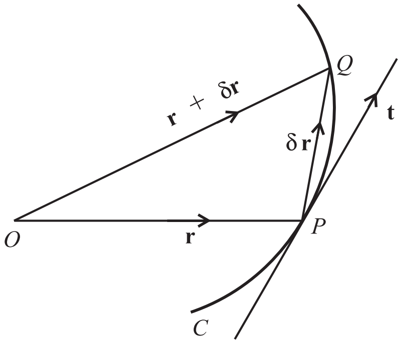
Similarly $\mathbf{r} = at^2 \mathbf{i} + 2at\mathbf{j} + 0\mathbf{k}$ is the vector equation of a parabola.

The terms **skew, twisted** or **tortuous** are often used for curves in a space.

11 Geometrical Significance of $d\mathbf{r}/dt$ and Unit Tangent Vector to a Curve

Geometrical Significance of $d\mathbf{r}/dt$:

Let $\mathbf{r} = \mathbf{f}\ (t)$ be the vector equation of a curve in space. Let $\mathbf{r}$ and $\mathbf{r} + \delta\mathbf{r}$ be the position vectors of two neighbouring points P and Q on this curve. Thus we have



$$\overrightarrow{OP} = \mathbf{r} = \mathbf{f}(t)$$

and $$\overrightarrow{OQ} = \mathbf{r} + \delta\mathbf{r} = \mathbf{f}(t + \delta t).$$

$$\therefore \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= (\mathbf{r} + \delta\mathbf{r}) - \mathbf{r} = \delta\mathbf{r}.$$

Thus $\dfrac{\delta\mathbf{r}}{dt}$ is a vector parallel to the chord PQ .

As $Q \to P$ *i.e.*, as $\delta t \to 0$, chord $PQ \to$ tangent at P to the curve.

$\therefore \quad \lim\limits_{\delta t \to 0} \dfrac{\delta\mathbf{r}}{\delta t} = \dfrac{d\mathbf{r}}{dt}$ is a vector parallel to the tangent at P to the curve $\mathbf{r} = \mathbf{f}\ (t)$.

Unit tangent vector to a curve:

Suppose in place of the scalar parameter t , we take the parameter as s where s denotes the arc length measured along the curve from any convenient fixed point C on the curve. Thus arc $CP = s$ and arc $CQ = s + \delta s$.

In this case $\dfrac{d\mathbf{r}}{ds}$ will be a vector along the tangent at P to the curve and in the direction of s increasing. Also we have

$$\left|\frac{d\,\mathbf{r}}{ds}\right| = \lim_{\delta t \to 0}\left|\frac{\delta\,\mathbf{r}}{\delta s}\right| = \lim_{\delta t \to 0} \frac{|\delta\mathbf{r}\,|}{\text{arc } PQ} = \lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ} = 1\,.$$

Thus $\dfrac{d\mathbf{r}}{ds}$ is a unit vector along the tangent at P in the direction of s increasing. We denote it by $\mathbf{t}$. Note that $\mathbf{t}$ always points in the direction of motion along the curve.

12 Application to Velocity and Acceleration

Velocity: If the scalar variable t be the time and $\mathbf{r}$ be the position vector of a moving particle P with respect to the origin O, then $\delta\mathbf{r}$ is the displacement of the particle in time δt .

The vector $\dfrac{\delta\mathbf{r}}{\delta t}$ is the average velocity of the particle during the interval δt . If $\mathbf{v}$ represents the velocity vector of the particle at P, then $\mathbf{v} = \lim\limits_{\delta t \to 0} \dfrac{\delta\mathbf{r}}{\delta t} = \dfrac{d\mathbf{r}}{dt}$ ·

Since $\dfrac{d\mathbf{r}}{dt}$ is a vector along the tangent at P to the curve in which the particle is moving, therefore the direction of velocity is along the tangent.

Acceleration: If $\delta\mathbf{v}$ be the change in the velocity $\mathbf{v}$ during the time δt, then $\dfrac{\delta\mathbf{v}}{\delta t}$ is the average acceleration during that interval. If $\mathbf{a}$ represents the acceleration of the particle at time t, then

$$\mathbf{a} = \lim_{\delta t \to 0} \frac{\delta\mathbf{v}}{\delta t} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d^2\mathbf{r}}{dt^2}.$$

Illustrative Examples

Example 1: *If* $\mathbf{r} = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\mathbf{k}$, *find*

(i) $\dfrac{d\mathbf{r}}{dt}$, **(Kashi 2013)** *(ii)* $\dfrac{d^2\mathbf{r}}{dt^2}$, **(Meerut 2010B)**

(iii) $\left|\dfrac{d\mathbf{r}}{dt}\right|$, **(Kashi 2013)** *(iv)* $\left|\dfrac{d^2\mathbf{r}}{dt^2}\right|$. **(Meerut 2009)**

Solution: Since $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are constant vectors, therefore $\dfrac{d\mathbf{i}}{dt} = \mathbf{0}$ etc. Therefore

(i) $$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(\sin t)\,\mathbf{i} + \frac{d}{dt}(\cos t)\,\mathbf{j} + \frac{d}{dt}(t)\,\mathbf{k} = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \mathbf{k}.$$

(ii) $$\frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d}{dt}(\cos t)\,\mathbf{i} - \frac{d}{dt}(\sin t)\,\mathbf{j} + \frac{d\mathbf{k}}{dt}$$

$$= -\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{0} = -\sin t\,\mathbf{i} - \cos t\,\mathbf{j}.$$

(iii) $$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{[(\cos t)^2 + (-\sin t)^2 + (1)^2]} = \sqrt{2}.$$

(iv) $$\left|\frac{d^2\mathbf{r}}{dt^2}\right| = \sqrt{[(-\sin t)^2 + (-\cos t)^2]} = 1.$$

Example 2: *If* $\mathbf{a}$, $\mathbf{b}$ *are constant vectors,* ω *is a constant, and* $\mathbf{r}$ *is a vector function of the scalar variable t given by* $\mathbf{r} = \cos\omega t\,\mathbf{a} + \sin\omega t\,\mathbf{b}$, *show that*

(i) $\dfrac{d^2\mathbf{r}}{dt^2} + \omega^2\mathbf{r} = \mathbf{0}$ **(Rohilkhand 2006)**

(ii) $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} = \omega\mathbf{a} \times \mathbf{b}$. **(Bundelkhand 2004; Kanpur 05)**

Solution: Since $\mathbf{a}$, $\mathbf{b}$ are constant vectors, therefore

$$\frac{d\mathbf{a}}{dt} = \mathbf{0}, \quad \frac{d\mathbf{b}}{dt} = \mathbf{0}.$$

(i) $$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(\cos \omega t)\,\mathbf{a} + \frac{d}{dt}(\sin \omega t)\,\mathbf{b}$$

$$= -\,\omega \sin \omega t\,\mathbf{a} + \omega \cos \omega t\,\mathbf{b}.$$

$\therefore$ $$\frac{d^2\mathbf{r}}{dt^2} = -\,\omega^2 \cos \omega t\,\mathbf{a} - \omega^2 \sin \omega t\,\mathbf{b}$$

$$= -\,\omega^2 (\cos \omega t\mathbf{a} + \sin \omega t\mathbf{b}) = -\,\omega^2\,\mathbf{r}.$$

$\therefore$ $$\frac{d^2\mathbf{r}}{dt^2} + \omega^2\,\mathbf{r} = \mathbf{0}.$$

(ii) $$\mathbf{r} \times \frac{d\mathbf{r}}{dt} = (\cos \omega t\mathbf{a} + \sin \omega t\,\mathbf{b}) \times (-\,\omega \sin \omega t\mathbf{a} + \omega \cos \omega t\mathbf{b})$$

$$= \omega \cos^2 \omega t\,\mathbf{a} \times \mathbf{b} - \omega \sin^2 \omega t\,\mathbf{b} \times \mathbf{a} \qquad [\because\ \mathbf{a} \times \mathbf{a} = \mathbf{0},\ \mathbf{b} \times \mathbf{b} = \mathbf{0}]$$

$$= \omega \cos^2 \omega t\,\mathbf{a} \times \mathbf{b} + \omega \sin^2 \omega t\,\mathbf{a} \times \mathbf{b}$$

$$= \omega\,(\cos^2 \omega t + \sin^2 \omega t)\,\mathbf{a} \times \mathbf{b} = \omega\mathbf{a} \times \mathbf{b}.$$

Example 3: *If* $\mathbf{r} = a \cos t\,\mathbf{i} + a \sin t\,\mathbf{j} + at \tan \alpha\,\mathbf{k}$, *find*

$$\left|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right| \text{ and } \left[\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3}\right].$$

(Purvanchal 07, 10; Agra 14)

Solution: We have $\dfrac{d\mathbf{r}}{dt} = -\,a \sin t\,\mathbf{i} + a \cos t\,\mathbf{j} + a \tan \alpha\,\mathbf{k}$

$$\frac{d^2\mathbf{r}}{dt^2} = -\,a \cos t\,\mathbf{i} - a \sin t\,\mathbf{j}, \qquad \left[\because\ \frac{d\,\mathbf{k}}{dt} = \mathbf{0}\right]$$

$$\frac{d^3\mathbf{r}}{dt^3} = a \sin t\,\mathbf{i} - a \cos t\,\mathbf{j}.$$

$\therefore$ $$\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\,a \sin t & a \cos t & a \tan \alpha \\ -\,a \cos t & -\,a \sin t & 0 \end{vmatrix}$$

$$= a^2 \sin t \tan \alpha\,\mathbf{i} - a^2 \cos t \tan \alpha\,\mathbf{j} + a^2\,\mathbf{k}.$$

$\therefore$ $$\left|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right| = \sqrt{(a^4 \sin^2 t \tan^2 \alpha + a^4 \cos^2 t \tan^2 \alpha + a^4)}$$

$$= a^2 \sec \alpha.$$

Also $$\left[\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3}\right] = \left(\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right) \bullet \frac{d^3\mathbf{r}}{dt^3}$$

$$= (a^2 \sin t \tan \alpha\,\mathbf{i} - a^2 \cos t \tan \alpha\,\mathbf{j} + a^2\,\mathbf{k}) \bullet (a \sin t\,\mathbf{i} - a \cos t\,\mathbf{j})$$

$$= a^3 \sin^2 t \tan \alpha\,\mathbf{i} \bullet \mathbf{i} + a^3 \cos^2 t \tan \alpha\,\mathbf{j} \bullet \mathbf{j} \qquad [\because\ \mathbf{i} \bullet \mathbf{j} = 0 \text{ etc.}]$$

$$= a^3 \tan \alpha\,(\sin^2 t + \cos^2 t) \qquad [\because\ \mathbf{i} \bullet \mathbf{i} = 1 = \mathbf{j} \bullet \mathbf{j}]$$

$$= a^3 \tan \alpha.$$

Example 4: *If* $\dfrac{d\mathbf{u}}{dt} = \mathbf{w} \times \mathbf{u}, \dfrac{d\mathbf{v}}{dt} = \mathbf{w} \times \mathbf{v}$, *show that* $\dfrac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \mathbf{w} \times (\mathbf{u} \times \mathbf{v})$.

(Garhwal 2003; Bundelkhand 09)

Solution: We have

$$\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \frac{d\mathbf{u}}{dt} \times \mathbf{v} + \mathbf{u} \times \frac{d\mathbf{v}}{dt} = (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} + \mathbf{u} \times (\mathbf{w} \times \mathbf{v})$$

$$= (\mathbf{v} \bullet \mathbf{w})\,\mathbf{u} - (\mathbf{v} \bullet \mathbf{u})\,\mathbf{w} + (\mathbf{u} \bullet \mathbf{v})\,\mathbf{w} - (\mathbf{u} \bullet \mathbf{w})\,\mathbf{v}$$

$$= (\mathbf{v} \bullet \mathbf{w})\,\mathbf{u} - (\mathbf{u} \bullet \mathbf{w})\,\mathbf{v} \qquad [\because \ \mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}]$$

$$= (\mathbf{w} \bullet \mathbf{v})\,\mathbf{u} - (\mathbf{w} \bullet \mathbf{u})\,\mathbf{v} = \mathbf{w} \times (\mathbf{u} \times \mathbf{v}).$$

Example 5: *If* $\mathbf{R}$ *be a unit vector in the direction of* $\mathbf{r}$, *prove that*

$$\mathbf{R} \times \frac{d\mathbf{R}}{dt} = \frac{1}{r^2}\,\mathbf{r} \times \frac{d\mathbf{r}}{dt}, \text{ where } r = |\mathbf{r}|.$$

(Bundelkhand 2008, 11)

Solution: We have $\mathbf{r} = r\mathbf{R}$; so that $\mathbf{R} = \dfrac{1}{r}\mathbf{r}$.

$\therefore$
$$\frac{d\mathbf{R}}{dt} = \frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\mathbf{r}.$$

Hence
$$\mathbf{R} \times \frac{d\mathbf{R}}{dt} = \frac{1}{r}\mathbf{r} \times \left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\mathbf{r}\right)$$

$$= \frac{1}{r^2}\mathbf{r} \times \frac{d\mathbf{r}}{dt} - \frac{1}{r^3}\frac{dr}{dt}\mathbf{r} \times \mathbf{r}$$

$$= \frac{1}{r^2}\mathbf{r} \times \frac{d\mathbf{r}}{dt}. \qquad [\because \ \mathbf{r} \times \mathbf{r} = \mathbf{0}]$$

Example 6: *If* $\mathbf{r}$ *is a vector function of a scalar t and* $\mathbf{a}$ *is a constant vector, m a constant, differentiate the following with respect to t :*

(i) $\mathbf{r} \bullet \mathbf{a}$,
(ii) $\mathbf{r} \times \mathbf{a}$,
(iii) $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt}$,
(iv) $\mathbf{r} \bullet \dfrac{d\mathbf{r}}{dt}$,
(v) $\mathbf{r}^2 + \dfrac{1}{\mathbf{r}^2}$,
(vi) $m\left(\dfrac{d\mathbf{r}}{dt}\right)^2$,
(vii) $\dfrac{\mathbf{r} + \mathbf{a}}{\mathbf{r}^2 + \mathbf{a}^2}$,
(viii) $\dfrac{\mathbf{r} \times \mathbf{a}}{\mathbf{r} \bullet \mathbf{a}}$.

Solution: (i) Let $R = \mathbf{r} \bullet \mathbf{a}$. [Note that $\mathbf{r} \bullet \mathbf{a}$ is a scalar]

Then
$$\frac{dR}{dt} = \frac{d\mathbf{r}}{dt} \bullet \mathbf{a} + \mathbf{r} \bullet \frac{d\mathbf{a}}{dt}$$

$$= \frac{d\mathbf{r}}{dt} \bullet \mathbf{a} + \mathbf{r} \bullet \mathbf{0} \qquad \left[\because \ \frac{d\mathbf{a}}{dt} = \mathbf{0}, \text{ as } \mathbf{a} \text{ is constant}\right]$$

$$= \frac{d\mathbf{r}}{dt} \bullet \mathbf{a} + 0$$

$$= \frac{d\mathbf{r}}{dt} \bullet \mathbf{a}.$$

(ii) Let $\mathbf{R} = \mathbf{r} \times \mathbf{a}$.

Then $$\frac{d\mathbf{R}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{a} + \mathbf{r} \times \frac{d\mathbf{a}}{dt}$$

$$= \frac{d\mathbf{r}}{dt} \times \mathbf{a} + \mathbf{r} \times \mathbf{0} \qquad \left[\because \frac{d\mathbf{a}}{dt} = \mathbf{0}\right]$$

$$= \frac{d\mathbf{r}}{dt} \times \mathbf{a} + \mathbf{0} = \frac{d\mathbf{r}}{dt} \times \mathbf{a}.$$

(iii) Let $\mathbf{R} = \mathbf{r} \times \dfrac{d\mathbf{r}}{dt}$.

Then $$\frac{d\mathbf{R}}{dt} = \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} + \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}$$

$$= \mathbf{0} + \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} \qquad \left[\because \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} = \mathbf{0}\right]$$

$$= \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}.$$

(iv) Let $R = \mathbf{r} \bullet \dfrac{d\mathbf{r}}{dt}$.

Then $$\frac{dR}{dt} = \frac{d\mathbf{r}}{dt} \bullet \frac{d\mathbf{r}}{dt} + \mathbf{r} \bullet \frac{d^2\mathbf{r}}{dt^2}$$

$$= \left(\frac{d\mathbf{r}}{dt}\right)^2 + \mathbf{r} \bullet \frac{d^2\mathbf{r}}{dt^2}.$$

(v) Let $R = \mathbf{r}^2 + \dfrac{1}{\mathbf{r}^2}$.

Then $$\frac{dR}{dt} = \frac{d}{dt}(\mathbf{r}^2) + \frac{d}{dt}\left(\frac{1}{\mathbf{r}^2}\right)$$

$$= \frac{d}{dt}(r^2) + \frac{d}{dt}\left(\frac{1}{r^2}\right), \text{where } r = |\mathbf{r}|$$

$$= 2r\frac{dr}{dt} - \frac{2}{r^3}\frac{dr}{dt}.$$

(vi) Let $R = m\left(\dfrac{d\mathbf{r}}{dt}\right)^2$.

Then $$\frac{dR}{dt} = m\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2$$

$$= 2\,m\frac{d\mathbf{r}}{dt} \bullet \frac{d^2\mathbf{r}}{dt^2} \qquad \left[\text{Note: } \frac{d\mathbf{r}^2}{dt} = 2\,\mathbf{r} \bullet \frac{d\mathbf{r}}{dt}\right]$$

$$= 2\,m\frac{d\mathbf{r}}{dt} \bullet \frac{d^2\mathbf{r}}{dt^2}.$$

(vii) Let $\mathbf{R} = \dfrac{\mathbf{r} + \mathbf{a}}{\mathbf{r}^2 + \mathbf{a}^2}$.

Then $$\frac{d\mathbf{R}}{dt} = \frac{1}{(\mathbf{r}^2 + \mathbf{a}^2)} \frac{d}{dt} (\mathbf{r} + \mathbf{a}) + \left\{\frac{d}{dt}\left(\frac{1}{\mathbf{r}^2 + \mathbf{a}^2}\right)\right\} (\mathbf{r} + \mathbf{a})$$

[Note that $\mathbf{r}^2 + \mathbf{a}^2$ is a scalar]

$$= \frac{1}{\mathbf{r}^2 + \mathbf{a}^2}\left(\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{a}}{dt}\right) - \left\{\frac{1}{(\mathbf{r}^2 + \mathbf{a}^2)^2} \frac{d}{dt} (\mathbf{r}^2 + \mathbf{a}^2)\right\} (\mathbf{r} + \mathbf{a})$$

$$= \frac{1}{\mathbf{r}^2 + \mathbf{a}^2} \frac{d\mathbf{r}}{dt} - \frac{2\,\mathbf{r} \bullet \dfrac{d\mathbf{r}}{dt}}{(\mathbf{r}^2 + \mathbf{a}^2)^2} (\mathbf{r} + \mathbf{a})$$

$$\left[\because \frac{d\mathbf{a}}{dt} = \mathbf{0},\ \frac{d}{dt}\mathbf{r}^2 = 2\mathbf{r} \bullet \frac{d\mathbf{r}}{dt},\ \frac{d}{dt}\mathbf{a}^2 = 0\right]$$

(viii) Let $\mathbf{R} = \dfrac{\mathbf{r} \times \mathbf{a}}{\mathbf{r} \bullet \mathbf{a}}$.

Then $$\frac{d\mathbf{R}}{dt} = \frac{1}{\mathbf{r} \bullet \mathbf{a}} \frac{d}{dt} (\mathbf{r} \times \mathbf{a}) + \left\{\frac{d}{dt}\left(\frac{1}{\mathbf{r} \bullet \mathbf{a}}\right)\right\} (\mathbf{r} \times \mathbf{a})$$

[Note that $\mathbf{r} \bullet \mathbf{a}$ is a scalar quantity]

$$= \frac{1}{\mathbf{r} \bullet \mathbf{a}}\left(\frac{d\,\mathbf{r}}{dt} \times \mathbf{a} + \mathbf{r} \times \frac{d\,\mathbf{a}}{dt}\right) - \left\{\frac{1}{(\mathbf{r} \bullet \mathbf{a})^2} \frac{d}{dt} (\mathbf{r} \bullet \mathbf{a})\right\} (\mathbf{r} \times \mathbf{a})$$

$$= \frac{\dfrac{d\mathbf{r}}{dt} \times \mathbf{a}}{\mathbf{r} \bullet \mathbf{a}} - \left\{\frac{1}{(\mathbf{r} \bullet \mathbf{a})^2}\left(\frac{d\mathbf{r}}{dt} \bullet \mathbf{a} + \mathbf{r} \bullet \frac{d\mathbf{a}}{dt}\right)\right\} (\mathbf{r} \times \mathbf{a})$$

$$= \frac{\dfrac{d\mathbf{r}}{dt} \times \mathbf{a}}{\mathbf{r} \bullet \mathbf{a}} - \frac{\dfrac{d\mathbf{r}}{dt} \bullet \mathbf{a}}{(\mathbf{r} \bullet \mathbf{a})^2} (\mathbf{r} \times \mathbf{a}). \qquad \left[\because \frac{d\mathbf{a}}{dt} = \mathbf{0}\right]$$

Example 7: *Find*

(i) $\dfrac{d}{dt}\left[\mathbf{r}, \dfrac{d\mathbf{r}}{dt}, \dfrac{d^2\mathbf{r}}{dt^2}\right]$; *(ii)* $\dfrac{d^2}{dt^2}\left[\mathbf{r}, \dfrac{d\mathbf{r}}{dt}, \dfrac{d^2\mathbf{r}}{dt^2}\right]$;

(iii) $\dfrac{d}{dt}\left[\mathbf{r} \times \left(\dfrac{d\mathbf{r}}{dt} \times \dfrac{d^2\mathbf{r}}{dt^2}\right)\right]$.

(Kumaun 2010)

Solution: (i) Let $R = \left[\mathbf{r}, \dfrac{d\mathbf{r}}{dt}, \dfrac{d^2\mathbf{r}}{dt^2}\right]$. Then R is the scalar triple product of three vectors $\mathbf{r}$, $\dfrac{d\mathbf{r}}{dt}$ and $\dfrac{d^2\mathbf{r}}{dt^2}$. Therefore using the rule for finding the derivative of a scalar triple product, we have

$$\frac{dR}{dt}=\left[\frac{d\mathbf{r}}{dt},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]+\left[\mathbf{r},\frac{d^2\mathbf{r}}{dt^2},\frac{d^2\mathbf{r}}{dt^2}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right]$$

$$=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right],$$

since scalar triple products having two equal vectors vanish.

(ii) Let $R=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]\cdot$ Then as in part (i)

$$\frac{dR}{dt}=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right].$$

Differentiating again, we get

$$\frac{d^2R}{dt^2}=\left[\frac{d\mathbf{r}}{dt},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d^2\mathbf{r}}{dt^2},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^4\mathbf{r}}{dt^4}\right]$$

$$=\left[\mathbf{r},\frac{d^2\mathbf{r}}{dt^2},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^4\mathbf{r}}{dt^4}\right].$$

(iii) Let $\mathbf{R}=\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)\cdot$

Then **R** is the vector triple product of three vectors. Therefore using the rule for finding the derivative of a vector triple product, we have

$$\frac{d\mathbf{R}}{dt}=\frac{d\mathbf{r}}{dt}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right)$$

$$=\frac{d\mathbf{r}}{dt}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right),$$

since $\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^2\mathbf{r}}{dt^2}=\mathbf{0}$, being vector product of two equal vectors.

Example 8: *If* $\mathbf{a}=\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j}+\theta\mathbf{k}$, $\mathbf{b}=\cos\theta\,\mathbf{i}-\sin\theta\,\mathbf{j}-3\mathbf{k}$, *and* $\mathbf{c}=2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$, *find* $\frac{d}{d\theta}\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\}$ *at* $\theta=\frac{\pi}{2}\cdot$ **(Kumaun 2010)**

Solution: We have $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta & -\sin\theta & -3\\ 2 & 3 & -3\end{vmatrix}$

$$=(3\sin\theta+9)\,\mathbf{i}+(3\cos\theta-6)\,\mathbf{j}+(3\cos\theta+2\sin\theta)\,\mathbf{k}\,.$$

$$\therefore\qquad \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \sin\theta & \cos\theta & \theta\\ 3\sin\theta+9 & 3\cos\theta-6 & 3\cos\theta+2\sin\theta\end{vmatrix}$$

$$= (3\cos^2\theta + 2\sin\theta\cos\theta - 3\theta\cos\theta + 6\theta)\,\mathbf{i}$$

$$+ (3\theta\sin\theta + 9\theta - 3\sin\theta\cos\theta - 2\sin^2\theta)\,\mathbf{j}$$

$$+ (-6\sin\theta - 9\cos\theta)\,\mathbf{k}\,.$$

$$\therefore \qquad \frac{d}{d\theta}\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\} = (-6\cos\theta\sin\theta + 2\cos^2\theta - 2\sin^2\theta$$

$$-3\cos\theta + 3\theta\sin\theta + 6)\,\mathbf{i} + (3\sin\theta + 3\theta\cos\theta + 9$$

$$-3\cos^2\theta + 3\sin^2\theta - 4\sin\theta\cos\theta)\,\mathbf{j}$$

$$+ (-6\cos\theta + 9\sin\theta)\,\mathbf{k}\,.$$

Putting $\theta = \pi/2$, we get the required derivative

$$= \left(4 + \frac{3}{2}\pi\right)\mathbf{i} + 15\mathbf{j} + 9\mathbf{k}\,.$$

Example 9: *A particle moves along the curve $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the velocity and acceleration at time $t = 0$ and $t = \frac{1}{2}\pi$. Find also the magnitudes of the velocity and acceleration at any time t.* **(Bundelkhand 2007, 14)**

Solution: Let $\mathbf{r}$ be the position vector of the particle at time t.

Then $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + 6t\,\mathbf{k}$. If $\mathbf{v}$ is the velocity of the particle at time t and $\mathbf{a}$ its acceleration at that time, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -4\sin t\,\mathbf{i} + 4\cos t\,\mathbf{j} + 6\mathbf{k}$$

and $$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -4\cos t\,\mathbf{i} - 4\sin t\,\mathbf{j}.$$

Magnitude of the velocity at time $t = |\mathbf{v}|$

$$= \sqrt{(16\sin^2 t + 16\cos^2 t + 36)} = \sqrt{(52)} = 2\sqrt{(13)}.$$

Magnitude of the acceleration $= |\mathbf{a}| = \sqrt{(16\cos^2 t + 16\sin^2 t)} = 4.$

At $t = 0$, $\mathbf{v} = 4\,\mathbf{j} + 6\,\mathbf{k}$, $\mathbf{a} = -4\,\mathbf{i}$.

At $t = \frac{1}{2}\pi$, $\mathbf{v} = -4\,\mathbf{i} + 6\,\mathbf{k}$, $\mathbf{a} = -4\,\mathbf{j}$.

Example 10: *A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$, where t is the time. Find the components of its velocity and acceleration at $t = 1$ in the direction* **i + j + 3k .** **(Purvanchal 2012)**

Solution: If $\mathbf{r}$ is the position vector of any point (x, y, z) on the given curve, then

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (t^3 + 1)\,\mathbf{i} + t^2\,\mathbf{j} + (2t + 5)\,\mathbf{k}\,.$$

Velocity $= \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = 3t^2\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k} = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ at $t = 1$.

Acceleration $= \mathbf{a} = \dfrac{d^2\mathbf{r}}{dt^2} = \dfrac{d}{dt}\left(\dfrac{d\mathbf{r}}{dt}\right) = 6t\mathbf{i} + 2\mathbf{j} = 6\mathbf{i} + 2\mathbf{j}$ at $t = 1$.

Now the unit vector in the given direction $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

$$= \frac{\mathbf{i}+\mathbf{j}+3\mathbf{k}}{|\mathbf{i}+\mathbf{j}+3\mathbf{k}|} = \frac{\mathbf{i}+\mathbf{j}+3\mathbf{k}}{\sqrt{(11)}} = \mathbf{b}, \text{ say.}$$

$\therefore$ the component of velocity in the given direction

$$= \mathbf{v} \bullet \mathbf{b} = \frac{(3\mathbf{i}+2\mathbf{j}+2\mathbf{k}) \bullet (\mathbf{i}+\mathbf{j}+3\mathbf{k})}{\sqrt{(11)}} = \frac{11}{\sqrt{(11)}} = \sqrt{(11)};$$

and the component of acceleration in the given direction

$$= \mathbf{a} \bullet \mathbf{b} = \frac{(6\mathbf{i}+2\mathbf{j}) \bullet (\mathbf{i}+\mathbf{j}+3\mathbf{k})}{\sqrt{(11)}} = \frac{8}{\sqrt{(11)}}.$$

Comprehensive Exercise 1

1. (i) If $\mathbf{r}$ is the position vector of a moving point and r is the modulus of $\mathbf{r}$, show that $\mathbf{r} \bullet \frac{d\mathbf{r}}{dt} = r\frac{dr}{dt}$. Interpret the relations $\mathbf{r} \bullet \frac{d\mathbf{r}}{dt} = 0$ and $\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \mathbf{0}$.

   (ii) If $\mathbf{r} \times d\mathbf{r} = \mathbf{0}$, show that $\hat{\mathbf{r}} =$ constant. **(Garhwal 2001; Meerut 05B; Purvanchal 12; Bundelkhand 12; Avadh 13)**

2. (i) If $\mathbf{r} = (t+1)\,\mathbf{i} + (t^2+t+1)\,\mathbf{j} + (t^3+t^2+t+1)\,\mathbf{k}$, find $\frac{d\mathbf{r}}{dt}$ and $\frac{d^2\mathbf{r}}{dt^2}$.

   (ii) If $\mathbf{r} = t^2\mathbf{i} - t\,\mathbf{j} + (2t+1)\,\mathbf{k}$, find at $t = 0$, the values of

   $\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \left|\frac{d\mathbf{r}}{dt}\right|, \left|\frac{d^2\mathbf{r}}{dt^2}\right|$. **(Kanpur 2002; Bundelkhand 2010, 13)**

3. Show that $\hat{\mathbf{r}} \times d\hat{\mathbf{r}} = (\mathbf{r} \times d\,\mathbf{r})/r^2$, where $\mathbf{r} = r\,\hat{\mathbf{r}}$.

4. (i) The position vector of a moving particle at time t is given by $\mathbf{r} = (3t-4)\,\mathbf{i} + (t^2-2)\,\mathbf{j} + 4t^3\,\mathbf{k}$. Find its velocity and acceleration at time $t = 2$. **(Kanpur 2005; Bundelkhand 13)**

   (ii) If $\mathbf{r} = t^3\,\mathbf{i} + \left(2t^3 - \frac{1}{5t^2}\right)\mathbf{j}$, show that $\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \mathbf{k}$. **(Agra 2007)**

   (iii) If $\mathbf{r} = (\cos nt)\,\mathbf{i} + (\sin nt)\,\mathbf{j}$, where n is a constant and t varies, show that $\mathbf{r} \times \frac{d\mathbf{r}}{dt} = n\mathbf{k}$. **(Garhwal 2002)**

5. (i) If $\mathbf{r} = (\sinh t)\,\mathbf{a} + (\cosh t)\,\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, then show that $\frac{d^2\mathbf{r}}{dt^2} = \mathbf{r}$.

   (ii) If $\mathbf{u} = t^2\mathbf{i} - t\mathbf{j} + (2t+1)\,\mathbf{k}$ and $\mathbf{v} = (2t-3)\,\mathbf{i} + \mathbf{j} - t\mathbf{k}$, find $\frac{d}{dt}(\mathbf{u} \bullet \mathbf{v})$, when $t = 1$. **(Bundelkhand 2007)**

6. (i) If $\mathbf{r} = e^{nt}\ \mathbf{a} + e^{-nt}\ \mathbf{b}$, where $\mathbf{a}$, $\mathbf{b}$ are constant vectors, show that

$$\frac{d^2\mathbf{r}}{dt^2} - n^2\mathbf{r} = \mathbf{0}.$$ **(Kumaun 2000; Rohilkhand 07; Agra 07; Kashi 14)**

(ii) Show that $\mathbf{r} = \mathbf{a}\, e^{mt} + \mathbf{b}\, e^{nt}$, where $\mathbf{a}$ and $\mathbf{b}$ are the constant vectors, is the solution of the differential equation $\dfrac{d^2\mathbf{r}}{dt^2} - (m+n)\dfrac{d\mathbf{r}}{dt} + mn\,\mathbf{r} = \mathbf{0}$.

Hence solve the equation

$$\frac{d^2\mathbf{r}}{dt^2} - \frac{d\mathbf{r}}{dt} - 2\mathbf{r} = \mathbf{0}, \quad \text{where } \mathbf{r} = \mathbf{i} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{j} \text{ for } t = 0.$$

(iii) If $\mathbf{r} = \mathbf{a}\sin\omega t + \mathbf{b}\cos\omega t + \dfrac{\mathbf{c}\,t}{\omega^2}\sin\omega t$, prove that

$$\frac{d^2\mathbf{r}}{dt^2} + \omega^2\mathbf{r} = \frac{2\mathbf{c}}{\omega}\cos\omega t,$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are constant vectors and ω is a constant scalar.

7. (i) A particle moves along the curve $x = e^{-t}, y = 2\cos 3t, z = 2\sin 3t$.

Determine the velocity and acceleration at any time t and their magnitudes at $t = 0$. **(Bundelkhand 2005; Rohilkhand 08)**

(ii) Show that if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are constant vectors, then $\mathbf{r} = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$ is the path of a particle moving with constant acceleration.

8. A particle moves so that its position vector is given by $\mathbf{r} = \cos\omega t\ \mathbf{i} + \sin\omega t\ \mathbf{j}$ where ω is a constant; show that

(i) the velocity of the particle is perpendicular to $\mathbf{r}$,

(ii) the acceleration is directed towards the origin and has magnitude proportional to the distance from the origin.

9. If $\mathbf{A} = 5t^2\ \mathbf{i} + t\ \mathbf{j} - t^3\ \mathbf{k}$ and $\mathbf{B} = \sin t\ \mathbf{i} - \cos t\ \mathbf{j}$, find

(i) $\dfrac{d}{dt}(\mathbf{A}\bullet\mathbf{B})$; (ii) $\dfrac{d}{dt}(\mathbf{A}\times\mathbf{B})$; (iii) $\dfrac{d}{dt}(\mathbf{A}\bullet\mathbf{A})$. **(Agra 2005)**

10. Prove the following :

(i) $$\frac{d}{dt}\left[\mathbf{a}\bullet\frac{d\mathbf{b}}{dt} - \frac{d\mathbf{a}}{dt}\bullet\mathbf{b}\right] = \mathbf{a}\bullet\frac{d^2\mathbf{b}}{dt^2} - \frac{d^2\mathbf{a}}{dt^2}\bullet\mathbf{b}$$

(ii) $$\frac{d}{dt}\left[\mathbf{a}\times\frac{d\mathbf{b}}{dt} - \frac{d\mathbf{a}}{dt}\times\mathbf{b}\right] = \mathbf{a}\times\frac{d^2\mathbf{b}}{dt^2} - \frac{d^2\mathbf{a}}{dt^2}\times\mathbf{b}$$

11. If $\mathbf{r}$ is a unit vector, then prove that

$$\left|\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right| = \left|\frac{d\mathbf{r}}{dt}\right|.$$

(Purvanchal 2006)

12. If $\mathbf{r}$ is a vector function of a scalar t, r its module, and $\mathbf{a}$, $\mathbf{b}$ are constant vectors, differentiate the following with respect to t :

(i) $r^3\,\mathbf{r} + \mathbf{a} \times \dfrac{d\,\mathbf{r}}{dt}$, (ii) $r^2\,\mathbf{r} + (\mathbf{a} \bullet \mathbf{r})\,\mathbf{b}$,

(iii) $r^n\,\mathbf{r}$, (iv) $(a\mathbf{r} + r\,\mathbf{b})^2$.

13. Find the unit tangent vector to any point on the curve

$$x = a\cos t\,,\ y = a\sin t\,,\ z = bt\,.$$

14. If the direction of a differentiable vector function $\mathbf{r}\ (t)$ is constant, show that

$$\mathbf{r} \times \left(\frac{d\,\mathbf{r}}{dt}\right) = \mathbf{0}. \qquad \textit{Or}$$

If $\mathbf{r}\ (t)$ is a vector of constant direction, show that its derivative is collinear with it.

15. If $\mathbf{e}$ is the unit vector making an angle θ with x-axis, show that $d\mathbf{e}/d\theta$ is a unit vector obtained by rotating $\mathbf{e}$ through a right angle in the direction of θ increasing.

Answers 1

2. (i) $\mathbf{i} + (2t + 1)\,\mathbf{j} + (3t^2 + 2t + 1)\,\mathbf{k}\,; 2\mathbf{j} + (6t + 2)\,\mathbf{k}$

(ii) $-\,\mathbf{j} + 2\mathbf{k};\ 2i;\ \sqrt{5};\ 2$

4. (i) $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 48\mathbf{k},\ \mathbf{a} = 2\mathbf{j} + 48\mathbf{k}$

5. (i) $\mathbf{r}$ (ii) $6t^2 - 10t - 2\,; -6$

7. (i) $\sqrt{37};\ \sqrt{(325)}$

9. (i) $(5t^2 - 1)\cos t + 11t\sin t$

(ii) $t^2\,(t\sin t - 3\cos t)\,\mathbf{i} - t^2\,(t\cos t + 3\sin t)\,\mathbf{j}$

$$-\,(11t\cos t - 5t^2\sin t + \sin t)\,\mathbf{k}$$

(iii) $100\,t^3 + 2\,t + 6t^5$

12. (i) $3r^2\,\dfrac{d\,r}{dt}\,\mathbf{r} + r^3\,\dfrac{d\,\mathbf{r}}{dt} + \mathbf{a} \times \dfrac{d^2\,\mathbf{r}}{dt^2}$

(ii) $2\,r\,\dfrac{dr}{dt}\,\mathbf{r} + r^2\,\dfrac{d\,\mathbf{r}}{dt} + \left(\mathbf{a} \bullet \dfrac{d\,\mathbf{r}}{dt}\right)\mathbf{b}$

(iii) $\left(nr^{n-1}\,\dfrac{dr}{dt}\right)\mathbf{r} + r^n\,\dfrac{d\,\mathbf{r}}{dt}$

(iv) $2\,(a\,\mathbf{r} + r\,\mathbf{b}) \bullet \left(a\,\dfrac{d\,\mathbf{r}}{dt} + \dfrac{dr}{dt}\,\mathbf{b}\right)$

13. $\dfrac{1}{\sqrt{(a^2 + b^2)}}\,(-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\,\mathbf{k})$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If $\mathbf{r} = \mathbf{a}e^{\omega t} + \mathbf{b}e^{-\omega t}$, where $\mathbf{a}$, $\mathbf{b}$ are constant vectors then $\dfrac{d^2\mathbf{r}}{dt^2} - \omega^2\mathbf{r}$ is equal to

(a) 1 (b) $\mathbf{0}$

(c) 2 (d) none of these

(Bundelkhand 2001; Agra 06)

2. A particle moves along the curve $\mathbf{r} = e^{-t}\cos t\,\mathbf{i} + e^{-t}\sin t\,\mathbf{j} + e^{-t}\mathbf{k}$. The magnitude of its velocity at $t = 0$ is

(a) $2\sqrt{3}$ (b) $\sqrt{3}/2$

(c) $\sqrt{3}$ (d) none of these

3. If $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, then $\dfrac{d^2\mathbf{r}}{dt^2} = \ldots\ldots.$

(a) $\mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z$ (b) $\mathbf{i}\dfrac{dx}{dt} + \mathbf{j}\dfrac{dy}{dt} + \mathbf{k}\dfrac{dz}{dt}$

(c) $\mathbf{i}\dfrac{d^2x}{dt^2} + \mathbf{j}\dfrac{d^2y}{dt^2} + \mathbf{k}\dfrac{d^2z}{dt^2}$ (d) none of these

(Kumaun 2011)

4. If $\mathbf{r} = (\cos nt)\,\mathbf{i} + (\sin nt)\,\mathbf{j}$, then

(a) $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} = n\mathbf{i}$ (b) $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} = n\mathbf{j}$

(c) $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} = n\mathbf{k}$ (d) none of these

(Kumaun 2006)

5. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors then the value of $\dfrac{d}{dt}(\mathbf{a} \bullet \mathbf{b})$ is

(a) $\dfrac{d\mathbf{a}}{dt} \bullet \mathbf{b}$ (b) $\mathbf{a} \bullet \dfrac{d\mathbf{b}}{dt}$

(c) $\mathbf{a} \bullet \dfrac{d\mathbf{b}}{dt} + \dfrac{d\mathbf{a}}{dt} \bullet \mathbf{b}$ (d) $\mathbf{a} \times \dfrac{d\mathbf{b}}{dt} + \dfrac{d\mathbf{a}}{dt} \times \mathbf{b}$

6. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors then the value of $\dfrac{d}{dt}(\mathbf{a} \times \mathbf{b})$ is

(a) $\mathbf{a} \times \dfrac{d\mathbf{b}}{dt}$ (b) $\dfrac{d\mathbf{a}}{dt} \times \mathbf{b}$

(c) $\mathbf{a} \times \dfrac{d\mathbf{b}}{dt} + \mathbf{b} \times \dfrac{d\mathbf{a}}{dt}$ (d) $\mathbf{a} \times \dfrac{d\mathbf{b}}{dt} + \dfrac{d\mathbf{a}}{dt} \times \mathbf{b}$

7. If $\mathbf{r} = 3\,\mathbf{i} - 6t^2\,\mathbf{j} + 4t\,\mathbf{k}$, then the value of $\dfrac{d\mathbf{r}}{dt}$ is

(a) $3t\,\mathbf{i} - 6t^2\,\mathbf{j} + 4t\mathbf{k}$ (b) $3\,\mathbf{i} - 12t\,\mathbf{j} + 4t\mathbf{k}$

(c) $-6t\,\mathbf{i} + 4\mathbf{k}$ (d) $-12t\,\mathbf{j} + 4\mathbf{k}$

8. Unit vector $\mathbf{a}$ and its derivative $\dfrac{d\mathbf{a}}{dt}$ are

(a) parallel (b) perpendicular

(c) along the same direction (d) none of these

9. If $\mathbf{r} = 3\mathbf{i} - t^2\,\mathbf{j} + 4t\mathbf{k}$, then $\left|\dfrac{d^2\,\mathbf{r}}{dt^2}\right|$ is equal to

(a) 0 (b) 1

(c) 2 (d) 3 **(Garhwal 2004, 13)**

10. The necessary and sufficient condition for the vector $\mathbf{a}(t)$ to have constant direction is

(a) $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0$ (b) $\mathbf{a} \times \dfrac{d\mathbf{a}}{dt} = 0$

(c) $\dfrac{d\mathbf{a}}{dt} = 0$ (d) none of these

(Garhwal 2006)

11. If $\mathbf{r} = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\mathbf{k}$, then the value of $\left|\dfrac{d^2\,\mathbf{r}}{dt^2}\right|$ is

(a) 1 (b) $\sqrt{2}$

(c) 2 (d) 0 **(Garhwal 2010, 14)**

12. A particle moves along the curve $x = e^{-t}, y = 2\cos 3t, z = 2\sin 3t$, then magnitude of velocity at $t = 0$ is

(a) $\sqrt{325}$ (b) $\sqrt{37}$

(c) $\sqrt{39}$ (d) none of these

(Garhwal 2011)

13. If $\mathbf{r} = (\sinh t)\,\mathbf{a} + (\cosh t)\,\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors then $\dfrac{d^2\,\mathbf{r}}{dt^2} =$

(a) $\mathbf{r}$ (b) $-\mathbf{r}$

(c) $\mathbf{0}$ (d) none of these

14. If $\mathbf{A} = 5t^2\,\mathbf{i} + t\,\mathbf{j} - t^3\,\mathbf{k}$, then $\dfrac{d}{dt}(\mathbf{A} \bullet \mathbf{A})$ is

(a) $(5t^3 - 1) + 11t$ (b) $10t^3 + 2t + 6t^5$

(c) $100t^3 + 2t + 6t^5$ (d) $100t^5 + 2t + 6t^3$

15. The necessary and sufficient condition for the vector function $\mathbf{a}(t)$ to be constant is

(a) $\frac{d\mathbf{a}}{dt} = \mathbf{0}$ (b) $\frac{da}{dt} = 0$

(c) $\mathbf{a} \bullet \frac{d\mathbf{a}}{dt} = 0$ (d) $\frac{d\mathbf{a}}{dt} \neq 0$

16. If $\mathbf{a}$ is a differential vector function of the scalar variable t and if $|\mathbf{a}| = a$, then $\frac{d}{dt}(\mathbf{a}^2)$ is

(a) $a\frac{da}{dt}$ (b) $\frac{da}{dt}$

(c) $2a\frac{da}{dt}$ (d) $\mathbf{a} \bullet \frac{da}{dt}$

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. If $\mathbf{r} = 3\mathbf{i} - 6t^2\,\mathbf{j} + 4t\mathbf{k}$, then $\frac{d\mathbf{r}}{dt} = \ldots\ldots$; $\frac{d^2\mathbf{r}}{dt^2} = \ldots\ldots$. **(Bundelkhand 2008)**

2. If $\mathbf{u} = t^2\,\mathbf{i} - t\,\mathbf{j} + (2t+1)\,\mathbf{k}$, $\mathbf{v} = (2t-3)\,\mathbf{i} + \mathbf{j} - t\,\mathbf{k}$, then $\frac{d}{dt}(\mathbf{u} \bullet \mathbf{v}) = \ldots\ldots$.

3. If $\mathbf{r} = (\cos \omega t)\,\mathbf{i} + (\sin \omega t)\,\mathbf{j}$, then $\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \ldots\ldots$.

4. The necessary and sufficient condition for the vector $\mathbf{a}(t)$ to have constant direction is ……

5. If $\mathbf{r} = 5\mathbf{i} + 3t^2\,\mathbf{j} + 2t\,\mathbf{k}$, then $\frac{d\mathbf{r}}{dt} = \ldots\ldots$; $\frac{d^2\mathbf{r}}{dt^2} = \ldots\ldots$. **(Bundelkhand 2010)**

6. $\frac{d}{dt}(\mathbf{a} \bullet \mathbf{b}) = \ldots\ldots$. **(Kumaun 2009)**

True or False

Write 'T' for true and 'F' for false statement.

1. A vector is said to be constant only if its magnitude is fixed and direction changes.

2. The necessary and sufficient condition for the vector $\mathbf{a}(t)$ to have constant magnitude is $\mathbf{a} \bullet \frac{d\mathbf{a}}{dt} = 0$.

Answers

Multiple Choice Questions

1.	(b)	**2.**	(c)	**3.**	(c)	**4.**	(c)	**5.**	(c)
6.	(d)	**7.**	(d)	**8.**	(b)	**9.**	(c)	**10.**	(a)
11.	(a)	**12.**	(b)	**13.**	(a)	**14.**	(c)	**15.**	(a)
16.	(c)								

Fill in the Blank(s)

1. $-12t\,\mathbf{j}+4\,\mathbf{k}\,;-12\,\mathbf{j}$
2. $6t^2-10t-2$
3. 0
4. $\mathbf{a}\times\frac{d\,\mathbf{a}}{dt}=\mathbf{0}$
5. $6t\,\mathbf{j}+2\,\mathbf{k}\,;6\,\mathbf{j}$
6. $\mathbf{a}\bullet\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\bullet\mathbf{b}$

True or False

1. *F*
2. *T*

☐

Chapter 9

Gradient, Divergence and Curl

1 Partial Derivatives of Vectors

Suppose $\mathbf{r}$ is a vector depending on more than one scalar variable. Let $\mathbf{r} = \mathbf{f}(x, y, z)$ *i.e.*, let $\mathbf{r}$ be a function of three scalar variables x, y and z. The partial derivative of $\mathbf{r}$ with respect to x is defined as

$$\frac{\partial \mathbf{r}}{\partial x} = \lim_{\delta x \to 0} \frac{\mathbf{f}(x + \delta x, y, z) - \mathbf{f}(x, y, z)}{\delta x}$$

if this limit exists. Thus $\partial \mathbf{r} / \partial x$ is nothing but the ordinary derivative of $\mathbf{r}$ with respect to x provided the other variables y and z are regarded as constants. Similarly we may define the partial derivatives $\dfrac{\partial \mathbf{r}}{\partial y}$ and $\dfrac{\partial \mathbf{r}}{\partial z}$.

Higher partial derivatives can also be defined as in Scalar Calculus. Thus, for example,

$$\frac{\partial^2 \mathbf{r}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial x}\right), \quad \frac{\partial^2 \mathbf{r}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{r}}{\partial y}\right), \quad \frac{\partial^2 \mathbf{r}}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \mathbf{r}}{\partial z}\right),$$

$$\frac{\partial^2 \mathbf{r}}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial y}\right), \quad \frac{\partial^2 \mathbf{r}}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{r}}{\partial x}\right).$$

If $\mathbf{r}$ has continuous partial derivatives of the second order at least, then, $\dfrac{\partial^2 \mathbf{r}}{\partial x\,\partial y} = \dfrac{\partial^2 \mathbf{r}}{\partial y\,\partial x}$ *i.e.*, the order of differentiation is immaterial. If $\mathbf{r} = \mathbf{f}\,(x, y, z)$, the total differential $d\mathbf{r}$ of $\mathbf{r}$ is given by $d\mathbf{r} = \dfrac{\partial \mathbf{r}}{\partial x}\,dx + \dfrac{\partial \mathbf{r}}{\partial y}\,dy + \dfrac{\partial \mathbf{r}}{\partial z}\,dz$.

2 The Vector Differential Operator Del (∇)

The vector differential operator ∇ (read as *del* or *nabla*) is defined as

$$\nabla \equiv \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k} \equiv \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}$$

and operates distributively.

The vector operator ∇ can generally be treated to behave as an ordinary vector. It possesses properties like ordinary vectors. The symbols $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}$ can be treated as its components along $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

3 Gradient of a Scalar Field

(Agra 2005)

Definition: *Let $f\,(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space (i.e., defines a differentiable scalar field). Then the gradient of f, written as ∇f or grad f, is defined as*

$$\nabla f = \left(\frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}\right) f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}\,.$$

It should be noted that ∇f is a vector whose three successive components are $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}\cdot$ Thus the gradient of a scalar field defines a vector field. *If f is a scalar point function, then ∇f is a vector point function.*

4 Formulas Involving Gradient

Theorem 1: Gradient of the sum of two scalar point functions. *If f and g are two scalar point functions, then*

$$\textit{grad}\,(f + g) = \textit{grad}\,f + \textit{grad}\,g \quad \textit{or} \quad \nabla\,(f + g) = \nabla f + \nabla g\,.$$

Proof: We have

$$\text{grad}\,(f + g) = \nabla\,(f + g) = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right)(f + g)$$

$$= \mathbf{i}\,\frac{\partial}{\partial x}(f+g) + \mathbf{j}\,\frac{\partial}{\partial y}(f+g) + \mathbf{k}\,\frac{\partial}{\partial z}(f+g)$$

$$= \mathbf{i}\,\frac{\partial f}{\partial x} + \mathbf{i}\,\frac{\partial g}{\partial x} + \mathbf{j}\,\frac{\partial f}{\partial y} + \mathbf{j}\,\frac{\partial g}{\partial y} + \mathbf{k}\,\frac{\partial f}{\partial z} + \mathbf{k}\,\frac{\partial g}{\partial z}$$

$$= \left(\mathbf{i}\,\frac{\partial f}{\partial x} + \mathbf{j}\,\frac{\partial f}{\partial y} + \mathbf{k}\,\frac{\partial f}{\partial z}\right) + \left(\mathbf{i}\,\frac{\partial g}{\partial x} + \mathbf{j}\,\frac{\partial g}{\partial y} + \mathbf{k}\,\frac{\partial g}{\partial z}\right)$$

$$= \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right) f + \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right) g$$

$$= \nabla f + \nabla g = \text{grad } f + \text{grad } g\,.$$

Similarly, we can prove that $\nabla(f - g) = \nabla f - \nabla g.$

Theorem 2: Gradient of a constant. *The necessary and sufficient condition for a scalar point function to be constant is that* $\nabla f = \mathbf{0}.$

Proof: If $f(x, y, z)$ is constant, then $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0.$

Therefore, $\text{grad } f = \mathbf{i}\,\frac{\partial f}{\partial x} + \mathbf{j}\,\frac{\partial f}{\partial y} + \mathbf{k}\,\frac{\partial f}{\partial z} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$

Hence the condition is necessary.

Conversely, let $\text{grad } f = \mathbf{0}$. Then $\mathbf{i}\,\frac{\partial f}{\partial x} + \mathbf{j}\,\frac{\partial f}{\partial y} + \mathbf{k}\,\frac{\partial f}{\partial z} = \mathbf{0}$.

Therefore, $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0.$

∴ f must be independent of x, y and z.

∴ f must be a constant. Hence the condition is sufficient.

Theorem 3: Gradient of the product of two scalar point functions.

If f and g are scalar point functions, then $\text{grad}(fg) = f \text{ grad } g + g \text{ grad } f$

or $\nabla(fg) = f\,\nabla g + g\,\nabla f.$ **(Kumaun 2014)**

Proof: We have $\nabla(fg) = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right)(fg)$

$$= \mathbf{i}\,\frac{\partial}{\partial x}(fg) + \mathbf{j}\,\frac{\partial}{\partial y}(fg) + \mathbf{k}\,\frac{\partial}{\partial z}(fg)$$

$$= \mathbf{i}\left(f\,\frac{\partial g}{\partial x} + g\,\frac{\partial f}{\partial x}\right) + \mathbf{j}\left(f\,\frac{\partial g}{\partial y} + g\,\frac{\partial f}{\partial y}\right) + \mathbf{k}\left(f\,\frac{\partial g}{\partial z} + g\,\frac{\partial f}{\partial z}\right)$$

$$= f\left(\mathbf{i}\,\frac{\partial g}{\partial x} + \mathbf{j}\,\frac{\partial g}{\partial y} + \mathbf{k}\,\frac{\partial g}{\partial z}\right) + g\left(\mathbf{i}\,\frac{\partial f}{\partial x} + \mathbf{j}\,\frac{\partial f}{\partial y} + \mathbf{k}\,\frac{\partial f}{\partial z}\right)$$

$$= f\,\nabla g + g\,\nabla f = f \text{ grad } g + g \text{ grad } f\,.$$

In particular, if c is a constant, then

$$\nabla(c\,f) = c\,\nabla f + f\,\nabla c = c\,\nabla f + \mathbf{0} = c\,\nabla f.$$

Theorem 4: Gradient of the quotient of two scalar functions. *If f and g are two scalar point functions, then* $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\,\nabla f - f\,\nabla g}{g^2}$.

(Kumaun 2013)

Proof: We have

$$\nabla\left(\frac{f}{g}\right) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left(\frac{f}{g}\right)$$

$$= \mathbf{i}\frac{\partial}{\partial x}\left(\frac{f}{g}\right) + \mathbf{j}\frac{\partial}{\partial y}\left(\frac{f}{g}\right) + \mathbf{k}\frac{\partial}{\partial z}\left(\frac{f}{g}\right).$$

But $\dfrac{\partial}{\partial x}\left(\dfrac{f}{g}\right) = \dfrac{g\dfrac{\partial f}{\partial x} - f\dfrac{\partial g}{\partial x}}{g^2}$, $\dfrac{\partial}{\partial y}\left(\dfrac{f}{g}\right) = \dfrac{g\dfrac{\partial f}{\partial y} - f\dfrac{\partial g}{\partial y}}{g^2}$,

and $\dfrac{\partial}{\partial z}\left(\dfrac{f}{g}\right) = \dfrac{g\dfrac{\partial f}{\partial z} - f\dfrac{\partial g}{\partial z}}{g^2}$.

$$\therefore \quad \nabla\left(\frac{f}{g}\right) = \frac{1}{g^2}\left\{\mathbf{i}\left(g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}\right) + \mathbf{j}\left(g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}\right) + \mathbf{k}\left(g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}\right)\right\}$$

$$= \frac{1}{g^2}\left\{g\left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right) - f\left(\mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j}\frac{\partial g}{\partial y} + \mathbf{k}\frac{\partial g}{\partial z}\right)\right\}$$

$$= \frac{1}{g^2}\{g\,\nabla f - f\,\nabla g\}.$$

Illustrative Examples

Example 1: *If* $\mathbf{A} = x^2 yz\,\mathbf{i} - 2xz^3\,\mathbf{j} + xz^2\,\mathbf{k}$, $\mathbf{B} = 2z\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$, *find the value of* $\dfrac{\partial^2}{\partial x\,\partial y}(\mathbf{A}\times\mathbf{B})$ *at* $(1, 0, -2)$.

Solution: We have $\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x^2 yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$

$$= (2x^3z^3 - xyz^2)\,\mathbf{i} + (2xz^3 + x^4 yz)\,\mathbf{j} + (x^2y^2z + 4xz^4)\,\mathbf{k}.$$

$$\therefore \quad \frac{\partial}{\partial y}(\mathbf{A}\times\mathbf{B}) = -xz^2\mathbf{i} + x^4 z\,\mathbf{j} + 2x^2yz\mathbf{k}.$$

Again $\dfrac{\partial^2}{\partial x\,\partial y}(\mathbf{A}\times\mathbf{B}) = \dfrac{\partial}{\partial x}\left\{\dfrac{\partial}{\partial y}(\mathbf{A}\times\mathbf{B})\right\} = -z^2\mathbf{i} + 4x^3z\mathbf{j} + 4xyz\,\mathbf{k}$. …(1)

Putting $x = 1$, $y = 0$ and $z = -2$ in (1), we get the required derivative at the point $(1, 0, -2) = -4\mathbf{i} - 8\mathbf{j}$.

Example 2: *If* $f(x, y, z) = 3x^2 y - y^3 z^2$, *find grad f at the point* $(1, -2, -1)$.

(Rohilkhand 2009)

Solution: We have

$$\text{grad } f = \nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(3x^2 y - y^3 z^2)$$

$$= \mathbf{i}\frac{\partial}{\partial x}(3x^2 y - y^3 z^2) + \mathbf{j}\frac{\partial}{\partial y}(3x^2 y - y^3 z^2)$$

$$+ \mathbf{k}\frac{\partial}{\partial z}(3x^2 y - y^3 z^2)$$

$$= \mathbf{i}(6xy) + \mathbf{j}(3x^2 - 3y^2 z^2) + \mathbf{k}(-2y^3 z)$$

$$= 6xy\,\mathbf{i} + (3x^2 - 3y^2 z^2)\,\mathbf{j} - 2y^3 z\,\mathbf{k}.$$

Putting $x = 1$, $y = -2$, $z = -1$, we get

$$\nabla f = 6(1)(-2)\,\mathbf{i} + \{3(1)^2 - 3(-2)^2(-1)^2\}\,\mathbf{j} - 2(-2)^3(-1)\,\mathbf{k}$$

$$= -12\,\mathbf{i} - 9\mathbf{j} - 16\mathbf{k}.$$

Example 3: *If* $r = |\mathbf{r}|$ *where* $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, *prove that*

(i) $\nabla f(r) = f'(r)\nabla r$, **(Rohilkhand 2008; Meerut 11)**

(ii) $\nabla r = \dfrac{1}{r}\mathbf{r}$,

(iii) $\nabla f(r) \times \mathbf{r} = \mathbf{0}$,

(iv) $\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\mathbf{r}}{r^3}$, **(Rohilkhand 2008, 09B)**

(v) $\nabla \log|\mathbf{r}| = \dfrac{\mathbf{r}}{r^2}$, **(Garhwal 2000; Kumaun 07; Rohilkhand 08; Bundelkhand 08; Meerut 11)**

(vi) $\nabla r^n = nr^{n-2}\,\mathbf{r}$. **(Rohilkhand 2008, 11; Bundelkhand 11)**

Solution: If $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, then $r = |\mathbf{r}| = \sqrt{(x^2 + y^2 + z^2)}$.

$\therefore$ $$r^2 = x^2 + y^2 + z^2.$$

(i) $$\nabla f(r) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f(r) = \mathbf{i}\frac{\partial}{\partial x} f(r) + \mathbf{j}\frac{\partial}{\partial y} f(r) + \mathbf{k}\frac{\partial}{\partial z} f(r)$$

$$= \mathbf{i}\,f'(r)\frac{\partial r}{\partial x} + \mathbf{j}\,f'(r)\frac{\partial r}{\partial y} + \mathbf{k}\,f'(r)\frac{\partial r}{\partial z} = f'(r)\left(\mathbf{i}\frac{\partial r}{\partial x} + \mathbf{j}\frac{\partial r}{\partial y} + \mathbf{k}\frac{\partial r}{\partial z}\right)$$

$$= f'(r)\nabla r.$$

(ii) We have $\nabla r = \mathbf{i}\dfrac{\partial r}{\partial x} + \mathbf{j}\dfrac{\partial r}{\partial y} + \mathbf{k}\dfrac{\partial r}{\partial z}\cdot$

Now $\quad r^2 = x^2 + y^2 + z^2.$

$$\therefore \quad 2r\frac{\partial r}{\partial x} = 2x \quad i.e., \quad \frac{\partial r}{\partial x} = \frac{x}{r}\cdot$$

Similarly, $\quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}\cdot$

$$\therefore \quad \nabla r = \frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k} = \frac{1}{r}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{1}{r}\mathbf{r} = \hat{\mathbf{r}}.$$

(iii) We have as in part (i), $\nabla f(r) = f'(r)\nabla r.$

But as in part (ii), $\nabla r = \dfrac{1}{r}\mathbf{r}.$

$$\therefore \quad \nabla f(r) = f'(r)\frac{1}{r}\mathbf{r}.$$

$$\therefore \quad \nabla f(r) \times \mathbf{r} = \left\{f'(r)\frac{1}{r}\mathbf{r}\right\} \times \mathbf{r} = \left\{\frac{1}{r}f'(r)\right\}(\mathbf{r} \times \mathbf{r}) = \mathbf{0}.$$

$[\because \ \mathbf{r} \times \mathbf{r} = \mathbf{0}]$

(iv) We have $\nabla\left(\dfrac{1}{r}\right) = \mathbf{i}\dfrac{\partial}{\partial x}\left(\dfrac{1}{r}\right) + \mathbf{j}\dfrac{\partial}{\partial y}\left(\dfrac{1}{r}\right) + \mathbf{k}\dfrac{\partial}{\partial z}\left(\dfrac{1}{r}\right)$

$$= \mathbf{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \mathbf{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \mathbf{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= -\frac{1}{r^2}\left(\frac{\partial r}{\partial x}\mathbf{i} + \frac{\partial r}{\partial y}\mathbf{j} + \frac{\partial r}{\partial z}\mathbf{k}\right)$$

$$= -\frac{1}{r^2}\left(\frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}\right) = -\frac{1}{r^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -\frac{1}{r^3}\mathbf{r}.$$

[see part (ii)]

(v) We have $\nabla \log|\mathbf{r}| = \nabla \log r$

$$= \mathbf{i}\frac{\partial}{\partial x}\log r + \mathbf{j}\frac{\partial}{\partial y}\log r + \mathbf{k}\frac{\partial}{\partial z}\log r$$

$$= \frac{1}{r}\frac{\partial r}{\partial x}\mathbf{i} + \frac{1}{r}\frac{\partial r}{\partial y}\mathbf{j} + \frac{1}{r}\frac{\partial r}{\partial z}\mathbf{k} = \frac{1}{r}\left(\frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}\right)$$

$$= \frac{1}{r^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{1}{r^2}\mathbf{r}.$$

(vi) We have $\nabla r^n = \mathbf{i}\dfrac{\partial}{\partial x}r^n + \mathbf{j}\dfrac{\partial}{\partial y}r^n + \mathbf{k}\dfrac{\partial}{\partial z}r^n$

$$= \mathbf{i}\,nr^{n-1}\frac{\partial r}{\partial x} + \mathbf{j}\,nr^{n-1}\frac{\partial r}{\partial y} + \mathbf{k}\,nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left(\mathbf{i}\frac{\partial r}{\partial x} + \mathbf{j}\frac{\partial r}{\partial y} + \mathbf{k}\frac{\partial r}{\partial z}\right) = nr^{n-1}\nabla r$$

$$= nr^{n-1}\,\frac{1}{r}\,\mathbf{r} \qquad \left[\because\ \nabla r = \frac{\mathbf{r}}{r} \text{ as in part (ii)}\right]$$

$$= nr^{n-2}\,\mathbf{r}.$$

Example 4: (i) *Interpret the symbol* $\mathbf{a}\bullet\nabla$ (ii) *Show that* $(\mathbf{a}\bullet\nabla)\,\phi = \mathbf{a}\bullet\nabla\,\phi$

(iii) *Show that* $(\mathbf{a}\bullet\nabla)\,\mathbf{r} = \mathbf{a}$. **(Kumaun 2008; Purvanchal 14)**

Solution: (i) Let $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$. Then

$$\mathbf{a}\bullet\nabla = (a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k})\bullet\left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right)$$

$$= a_1\,\frac{\partial}{\partial x} + a_2\,\frac{\partial}{\partial y} + a_3\,\frac{\partial}{\partial z}\,.$$

(ii) $$(\mathbf{a}\bullet\nabla)\,\phi = \left(a_1\,\frac{\partial}{\partial x} + a_2\,\frac{\partial}{\partial y} + a_3\,\frac{\partial}{\partial z}\right)\phi.$$

Also $$\mathbf{a}\bullet\nabla\phi = (a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k})\bullet\left(\frac{\partial\phi}{\partial x}\,\mathbf{i} + \frac{\partial\phi}{\partial y}\,\mathbf{j} + \frac{\partial\phi}{\partial z}\,\mathbf{k}\right)$$

$$= a_1\,\frac{\partial\phi}{\partial x} + a_2\,\frac{\partial\phi}{\partial y} + a_3\,\frac{\partial\phi}{\partial z}\,.$$

Hence $(\mathbf{a}\bullet\nabla)\,\phi = \mathbf{a}\bullet\nabla\phi.$

(iii) $$(\mathbf{a}\bullet\nabla)\,\mathbf{r} = \left(a_1\,\frac{\partial}{\partial x} + a_2\,\frac{\partial}{\partial y} + a_3\,\frac{\partial}{\partial z}\right)\mathbf{r} = a_1\,\frac{\partial\mathbf{r}}{\partial x} + a_2\,\frac{\partial\mathbf{r}}{\partial y} + a_3\,\frac{\partial\mathbf{r}}{\partial z}\,.$$

But $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\therefore \quad \frac{\partial\mathbf{r}}{\partial x} = \mathbf{i},\ \frac{\partial\mathbf{r}}{\partial y} = \mathbf{j},\ \frac{\partial\mathbf{r}}{\partial z} = \mathbf{k}.$$

$$\therefore \quad (\mathbf{a}\bullet\nabla)\,\mathbf{r} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k} = \mathbf{a}.$$

Comprehensive Exercise 1

1. If $\mathbf{F} = e^{xy}\,\mathbf{i} + (x - 2y)\,\mathbf{j} + x\sin y\,\mathbf{k}$, calculate

   (i) $\dfrac{\partial\mathbf{F}}{\partial x}$, (ii) $\dfrac{\partial\mathbf{F}}{\partial y}$, (iii) $\dfrac{\partial^2\mathbf{F}}{\partial x^2}$,

   (iv) $\dfrac{\partial^2\mathbf{F}}{\partial x\,\partial y}$, (v) $\dfrac{\partial^2\mathbf{F}}{\partial y^2}$.

2. If $\mathbf{f} = (2x^2 y - x^4)\,\mathbf{i} + (e^{xy} - y\sin x)\,\mathbf{j} + x^2\cos y\,\mathbf{k}$, verify that

   $$\frac{\partial^2\mathbf{f}}{\partial y\,\partial x} = \frac{\partial^2\mathbf{f}}{\partial x\,\partial y}\,.$$

3. If $\mathbf{u} = xyz\,\mathbf{i} + xz^2\,\mathbf{j} - y^3\,\mathbf{k}$ and $\mathbf{v} = x^3\,\mathbf{i} - xyz\,\mathbf{j} + x^2 z\,\mathbf{k}$, calculate

$$\frac{\partial^2 \mathbf{u}}{\partial y^2} \times \frac{\partial^2 \mathbf{v}}{\partial x^2} \text{ at the point } (1, 1, 0).$$

4. If $\phi(x, y, z) = x^2 y + y^2 x + z^2$, find $\nabla\phi$ at the point $(1, 1, 1)$.

5. Find grad f, where f is given by $f = x^3 - y^3 + xz^2$, at the point $(1, -1, 2)$.

6. If $\phi(x, y, z) = xy^2 z$ and $\mathbf{f} = xz\mathbf{i} - xy\mathbf{j} + yz^2\mathbf{k}$, show that

$$\frac{\partial^3}{\partial x^2 \partial z}(\phi\mathbf{f}) \text{ at } (2, -1, 1) \text{ is } 4\mathbf{i} + 2\mathbf{j}.$$

7. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, prove that

$$(\text{grad } u) \bullet [(\text{grad } v) \times (\text{grad } w)] = 0.$$ **(Meerut 2007B)**

8. If $\mathbf{F} = \left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)\mathbf{i} + \left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right)\mathbf{j} + \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right)\mathbf{k}$, prove that

(i) $\mathbf{F} = \mathbf{r} \times \nabla f$, (ii) $\mathbf{F} \bullet \mathbf{r} = 0$, (iii) $\mathbf{F} \bullet \nabla f = 0$.

9. If $\phi = (3r^2 - 4r^{1/2} + 6r^{-1/3})$, show that $\nabla\phi = 2(3 - r^{-3/2} - r^{-7/3})\,\mathbf{r}$.

10. Prove that $\nabla\phi \bullet d\mathbf{r} = d\phi$. **(Meerut 2005, 06, 09B; Kumaun 08)**

11. Prove that $f(u)\nabla u = \nabla \int f(u)\, du$. **(Kumaun 2012, 13)**

12. ρ and p are two scalar point functions such that ρ is a function of p; show that $\nabla\rho = \frac{d\rho}{dp}\nabla p$.

13. Show that $\frac{d\phi}{ds} = \nabla\phi \bullet \frac{d\mathbf{r}}{ds}$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and ϕ is a function of x, y and z.

14. Prove that $\mathbf{A} \bullet \left(\nabla \frac{1}{r}\right) = -\frac{\mathbf{A} \bullet \mathbf{r}}{r^3}$. **(Meerut 2010)**

15. Prove that $\nabla r^{-3} = -3r^{-5}\,\mathbf{r}$. **(Meerut 2009, 12)**

16. Show that

(i) grad $(\mathbf{r} \bullet \mathbf{a}) = \mathbf{a}$, **(Avadh 2010)**

(ii) grad $[\mathbf{r}, \mathbf{a}, \mathbf{b}] = \mathbf{a} \times \mathbf{b}$,

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

Answers 1

1. (i) $ye^{xy}\,\mathbf{i} + \mathbf{j} + \sin y\,\mathbf{k}$ (ii) $xe^{xy}\,\mathbf{i} - 2\,\mathbf{j} + x\cos y\,\mathbf{k}$

(iii) $y^2 e^{xy}\,\mathbf{i}$ (iv) $e^{xy}(xy + 1)\,\mathbf{i} + \cos y\,\mathbf{k}$

(v) $x^2 e^{xy}\,\mathbf{i} - x \sin y\,\mathbf{k}$

3. $-36\,\mathbf{j}$

4. $3\mathbf{i} + 3\mathbf{j} + 2\,\mathbf{k}$

5. $7\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

5 Equipotential Surfaces or Level Surfaces

Let $f(x, y, z)$ be a scalar field over a region R. The points satisfying an equation of the type

$$f(x, y, z) = c, \text{ (arbitrary constant)}$$

constitute a family of surfaces in three dimensional space. The surfaces of this family are called ***level surfaces***. Any surface of this family is such that the value of the function f at any point of it is the same. Therefore these surfaces are also called ***iso-f-surfaces.***

Theorem 1: *Let $f(x, y, z)$ be a scalar field over a region R. Then through any point of R there passes one and only one level surface.*

Proof: Let (x_1, y_1, z_1) be any point of the region R.

Then the level surface $f(x, y, z) = f(x_1, y_1, z_1)$ passes through this point.

Now suppose the level surfaces $f(x, y, z) = c_1$ and $f(x, y, z) = c_2$ pass through the point (x_1, y_1, z_1). Then

$$f(x_1, y_1, z_1) = c_1 \quad \text{and} \quad f(x_1, y_1, z_1) = c_2 .$$

Since $f(x, y, z)$ has a unique value at (x_1, y_1, z_1), therefore we have $c_1 = c_2$.

Hence only one level surface passes through the point (x_1, y_1, z_1).

Theorem 2: *∇f is a vector normal to the surface $f(x, y, z) = c$ where c is a constant.*

Proof: Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of any point $P(x, y, z)$ on the level surface $f(x, y, z) = c$. Let $Q(x + \delta x, y + \delta y, z + \delta z)$ be a neighbouring point on this surface.

Then the position vector of $Q = \mathbf{r} + \delta\mathbf{r} = (x + \delta x)\,\mathbf{i} + (y + \delta y)\,\mathbf{j} + (z + \delta z)\,\mathbf{k}$.

$$\therefore \qquad \overrightarrow{PQ} = (\mathbf{r} + \delta\mathbf{r}) - \mathbf{r} = \delta\mathbf{r} = \delta x\,\mathbf{i} + \delta y\,\mathbf{j} + \delta z\,\mathbf{k} .$$

As $Q \to P$, the line PQ tends to tangent at P to the level surface.

Therefore $d\,\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ lies in the tangent plane to the surface at P.

From the differential calculus, we have

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

$$= \left(\mathbf{i}\,\frac{\partial f}{\partial x} + \mathbf{j}\,\frac{\partial f}{\partial y} + \mathbf{k}\,\frac{\partial f}{\partial z}\right) \bullet (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = \nabla f \bullet d\,\mathbf{r}.$$

Since $f(x, y, z) =$ constant, therefore $df = 0$.

$\therefore\ \nabla f \bullet d\mathbf{r} = 0$ so that ∇f is a vector perpendicular to $d\,\mathbf{r}$ and therefore to the tangent plane at P to the surface $f(x, y, z) = c$.

Hence ∇f is a vector normal to the surface $f(x, y, z) = c$.

Thus if $f(x, y, z)$ is a scalar field defined over a region R, then ∇f at any point (x, y, z) is a vector in the direction of normal at that point to the level surface $f(x, y, z) = c$ passing through that point.

6 Directional Derivative of a Scalar Point Function

Definition: *Let $f(x, y, z)$ define a scalar field in a region R and let P be any point in this region. Suppose Q is a point in this region in the neighbourhood of P in the direction of a given unit vector $\hat{\mathbf{a}}$.*

Then $\lim_{Q \to P} \dfrac{f(Q) - f(P)}{PQ}$, if it exists, is called the ***directional derivative*** *of f at P in the direction of $\hat{\mathbf{a}}$.*

Interpretation of Directional Derivative: Let P be the point (x, y, z) and let Q be the point $(x + \delta x, y + \delta y, z + \delta z)$. Suppose $PQ = \delta s$. Then δs is a small element at P in the direction of $\hat{\mathbf{a}}$. If

$$\delta f = f(x + \delta x, y + \delta y, z + \delta z) - f(x, y, z) = f(Q) - f(P),$$

then $\dfrac{\delta f}{\delta s}$ represents the average rate of change of f per unit distance in the direction of $\hat{\mathbf{a}}$. Now the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$ is $\lim_{Q \to P} \dfrac{f(Q) - f(P)}{PQ} = \lim_{\delta s \to 0} \dfrac{\delta f}{\delta s} = \dfrac{df}{ds}$ · It represents the rate of change of f with respect to distance s at the point P in the direction of unit vector $\hat{\mathbf{a}}$.

Theorem 1: *The directional derivative of a scalar field f at a point $P(x, y, z)$ in the direction of a unit vector $\hat{\mathbf{a}}$ is given by $\dfrac{df}{ds} = \nabla f \bullet \hat{\mathbf{a}}$.*

Proof: Let $f(x, y, z)$ define a scalar field in the region R. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ denote the position vector of any point $P(x, y, z)$ in this region. If s denotes the distance of P from some fixed point A in the direction of $\hat{\mathbf{a}}$, then δs denotes small element at P in the direction of $\hat{\mathbf{a}}$. Therefore $\dfrac{d\mathbf{r}}{ds}$ is a unit vector at P in this direction *i.e.*, $\dfrac{d\mathbf{r}}{ds} = \hat{\mathbf{a}}$.

But $\qquad \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

$$\therefore \qquad \frac{d\,\mathbf{r}}{ds} = \frac{dx}{ds}\,\mathbf{i} + \frac{dy}{ds}\,\mathbf{j} + \frac{dz}{ds}\,\mathbf{k} = \hat{\mathbf{a}}.$$

Now

$$\nabla f \bullet \hat{\mathbf{a}} = \left(\mathbf{i}\,\frac{\partial f}{\partial x} + \mathbf{j}\,\frac{\partial f}{\partial y} + \mathbf{k}\,\frac{\partial f}{\partial z}\right) \bullet \left(\frac{dx}{ds}\,\mathbf{i} + \frac{dy}{ds}\,\mathbf{j} + \frac{dz}{ds}\,\mathbf{k}\right)$$

$$= \frac{\partial f}{dx}\,\frac{dx}{ds} + \frac{\partial f}{dy}\,\frac{dy}{ds} + \frac{\partial f}{dz}\,\frac{dz}{ds} = \frac{d\,f}{ds}$$

= directional derivative of f at P in the direction of $\hat{\mathbf{a}}$.

Alternative Proof: Let Q be a point in the neighbourhood of P in the direction of the given unit vector $\hat{\mathbf{a}}$. If l, m, n are the direction cosines of the line PQ, then $l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}$ = the unit vector in the direction of $PQ = \hat{\mathbf{a}}$. Further if $PQ = \delta s$, then the co-ordinates of Q are $(x + l\delta s, y + m\delta s, z + n\delta s)$. Now the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$ is

$$= \lim_{Q \to P} \frac{f\,(Q) - f\,(P)}{PQ}$$

$$= \lim_{\delta s \to 0} \frac{f\,(x + l\delta s\,,\, y + m\delta s\,,\, z + n\delta s) - f\,(x\,,\, y\,,\, z)}{\delta s}$$

$$= \lim_{\delta s \to 0} \frac{f\,(x\,,\, y\,,\, z) + \left(l\delta s\,\frac{\partial f}{\partial x} + m\delta s\,\frac{\partial f}{\partial y} + n\delta s\,\frac{\partial f}{\partial z}\right) + \ldots - f\,(x\,,\, y\,,\, z)}{\delta s},$$

on expanding by Taylor's theorem

$$= l\,\frac{\partial f}{\partial x} + m\,\frac{\partial f}{\partial y} + n\,\frac{\partial f}{\partial z}$$

$$= \left(\frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}\right) \bullet (l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}) = \nabla f \bullet \hat{\mathbf{a}}\,.$$

Theorem 2: *If $\hat{\mathbf{n}}$ be a unit vector normal to the level surface $f\,(x\,,\, y\,,\, z) = c$ at a point $P\,(x\,,\, y\,,\, z)$ and n be the distance of P from some fixed point A in the direction of $\hat{\mathbf{n}}$ so that δn represents element of normal at P in the direction of $\hat{\mathbf{n}}$, then grad $f = \dfrac{df}{dn}\,\hat{\mathbf{n}}$.*

Proof: We have grad $f = \nabla f = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} + \dfrac{\partial f}{\partial z}\,\mathbf{k}$.

Also grad f is a vector normal to the surface $f\,(x\,,\, y\,,\, z) = c$. Since $\hat{\mathbf{n}}$ is a unit vector normal to the surface $f\,(x\,,\, y\,,\, z) = c$, therefore let grad $f = A\,\hat{\mathbf{n}}$, where A is some scalar to be determined.

Now $\dfrac{df}{dn}$ = directional derivative of f in the direction of $\hat{\mathbf{n}}$

$$= \nabla f \bullet \hat{\mathbf{n}} = A\,\hat{\mathbf{n}} \bullet \hat{\mathbf{n}} = A. \qquad [\because\ \nabla f = \text{grad } f = A\,\hat{\mathbf{n}}\ ]$$

$$\therefore \quad \text{grad } f = \nabla f = \frac{df}{dn}\,\hat{\mathbf{n}}.$$

Note: If the vector $\hat{\mathbf{n}}$ is in the direction of f increasing, then $\dfrac{df}{dn}$ is positive. Therefore ∇f is a vector normal to the surface $f(x, y, z) = c$ in the direction of f increasing.

Theorem 3: *Grad f is a vector in the direction of which the maximum value of the directional derivative of f i.e.,* $\dfrac{df}{ds}$ *occurs.*

Proof: The directional derivative of f in the direction of $\hat{\mathbf{a}}$ is given by

$$\frac{df}{ds} = \nabla f \bullet \hat{\mathbf{a}} = \left(\frac{df}{dn}\,\hat{\mathbf{n}}\right) \bullet \hat{\mathbf{a}} \qquad \left[\because\ \nabla f = \frac{df}{dn}\,\hat{\mathbf{n}}\right]$$

$$= \frac{df}{dn}\,(\hat{\mathbf{n}} \bullet \hat{\mathbf{a}}) = \frac{df}{dn}\cos\theta,$$ where θ is the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{n}}$.

Now $\dfrac{df}{dn}$ is fixed. Therefore $\dfrac{df}{dn}\cos\theta$ is maximum when $\cos\theta$ is maximum *i.e.*, when $\cos\theta = 1$. But $\cos\theta$ will be 1 when the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{n}}$ is 0 *i.e.*, when $\hat{\mathbf{a}}$ is along the unit normal vector $\hat{\mathbf{n}}$.

Therefore the directional derivative is maximum along the normal to the surface. Its maximum value is $= \dfrac{df}{dn} = |\text{grad } f|$.

7 Tangent Plane and Normal to a Level Surface

Tangent plane to a surface: The tangent to any curve drawn on a surface is called a *tangent line* to the surface. All the tangent lines to a surface at the point P lie in a plane. This plane is called the *tangent plane* to the surface at the point P on it.

Normal to a surface: The normal to a surface at the point P is a straight line passing through P and perpendicular to the tangent plane at P.

Angle between two surfaces: The angle between the two surfaces at a point P is the angle between the normals to the two surfaces at that point.

Equations of the Tangent Plane and Normal to a Surface:

To find the equations of the tangent plane and normal to the surface $f(x, y, z) = c$.

Let $f(x, y, z) = c$ be the equation of a level surface. Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ be the position vector of any point $P(x, y, z)$ on this surface.

Then $\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$ is a vector along the normal to the surface at P *i.e.*, ∇f is perpendicular to the tangent plane at P.

Tangent plane at *P*: Let $\mathbf{R} = X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k}$ be the position vector of any current point $Q\,(X, Y, Z)$ on the tangent plane at P to the surface. The vector

$$\overrightarrow{PQ} = \mathbf{R} - \mathbf{r} = (X - x)\,\mathbf{i} + (Y - y)\,\mathbf{j} + (Z - z)\,\mathbf{k}$$

lies in the tangent plane at P. Therefore it is perpendicular to the vector ∇f.

$$\therefore \qquad (\mathbf{R} - \mathbf{r}) \bullet \nabla f = 0$$

or
$$[(X - x)\,\mathbf{i} + (Y - y)\,\mathbf{j} + (Z - z)\,\mathbf{k}] \bullet \left(\frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}\right) = 0$$

or
$$(X - x)\frac{\partial f}{\partial x} + (Y - y)\frac{\partial f}{\partial y} + (Z - z)\frac{\partial f}{\partial z} = 0, \qquad \ldots(1)$$

is the equation of the tangent plane at P.

Normal at *P*: Let $\mathbf{R} = X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k}$
be the position vector of any current point $Q\,(X, Y, Z)$ on the normal at P to the surface. The vector $\overrightarrow{PQ} = \mathbf{R} - \mathbf{r} = (X - x)\,\mathbf{i} + (Y - y)\,\mathbf{j} + (Z - z)\,\mathbf{k}$ lies along the normal at P to the surface. Therefore it is parallel to the vector ∇f.

$$\therefore \qquad (\mathbf{R} - \mathbf{r}) \times \nabla f = \mathbf{0} \qquad \ldots(2)$$

is the vector equation of the normal at P to the given surface.

Cartesian form: The vectors

$$(X - x)\,\mathbf{i} + (Y - y)\,\mathbf{j} + (Z - z)\,\mathbf{k} \text{ and } \nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k},$$

will be parallel if

$$(X - x)\,\mathbf{i} + (Y - y)\,\mathbf{j} + (Z - z)\,\mathbf{k} = p\left(\frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}\right),$$

where p is some scalar.
Equating the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, we get

$$X - x = p\frac{\partial f}{\partial x},\; Y - y = p\frac{\partial f}{\partial y},\; Z - z = p\frac{\partial f}{\partial z}$$

or
$$\frac{X - x}{\dfrac{\partial f}{\partial x}} = \frac{Y - y}{\dfrac{\partial f}{\partial y}} = \frac{Z - z}{\dfrac{\partial f}{\partial z}} \qquad \ldots(3)$$

which are the equations of the normal at P.

Note: The vector $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ is along the normal to the surface $F\,(x, y, z) = 0$ at the point (x, y, z).

Illustrative Examples

Example 5: *Find a unit normal vector to the level surface* $x^2y + 2xz = 4$ *at the point* $(2, -2, 3)$. **(Kashi 2014)**

Solution: The equation of the level surface is $f(x, y, z) \equiv x^2y + 2xz = 4$.

The vector grad f is along the normal to the surface at the point (x, y, z).

We have grad $f = \nabla(x^2y + 2xz) = (2xy + 2z)\,\mathbf{i} + x^2\,\mathbf{j} + 2x\,\mathbf{k}$.

$\therefore$ at the point $(2, -2, 3)$, grad $f = -2\,\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$.

$\therefore$ $-2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}$ is a vector along the normal to the given surface at the point $(2, -2, 3)$.

Hence a unit normal vector to the surface at this point

$$= \frac{-2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}}{|-2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}|} = \frac{-2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}}{\sqrt{(4 + 16 + 16)}} = -\frac{1}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}.$$

The vector $-\left(-\frac{1}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}\right)$ *i.e.*, $\frac{1}{3}\,\mathbf{i} - \frac{2}{3}\,\mathbf{j} - \frac{2}{3}\,\mathbf{k}$ is also a unit normal vector to the given surface at the point $(2, -2, 3)$.

Alternate Solution: The given surface is

$$\phi(x, y, z) \equiv x^2y + 2xz - 4 = 0.$$

We have $\dfrac{\partial\phi}{\partial x} = 2xy + 2z,\ \dfrac{\partial\phi}{\partial y} = x^2,\ \dfrac{\partial\phi}{\partial z} = 2z.$

$\therefore$ At the point $(2, -2, 3)$, we have

$$\frac{\partial\phi}{\partial x} = -2,\ \frac{\partial\phi}{\partial y} = 4,\ \frac{\partial\phi}{\partial z} = 4.$$

$\therefore$ A vector along the normal to the given surface at the point $(2, -2, 3)$

$$= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (-2, 4, 4) \quad \text{or} \quad (-1, 2, 2).$$

We have $|(-1, 2, 2)| = \sqrt{(-1)^2 + 2^2 + 2^2} = 3.$

Hence, a unit normal vector to the given surface at the point $(2, -2, 3)$

$$= \frac{1}{3}(-1, 2, 2) = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right).$$

Example 6: *Find the directional derivative of* $f(x, y, z) = x^2yz + 4xz^2$ *at the point* $(1, -2, -1)$ *in the direction of the vector* $2\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}$.

(Bundelkhand 2007; Rohilkhand 11; Kashi 13; Agra 14)

Solution: We have $f(x, y, z) = x^2yz + 4xz^2$.

$$\therefore \quad \text{grad } f = (2xyz + 4z^2)\,\mathbf{i} + x^2z\mathbf{j} + (x^2y + 8xz)\,\mathbf{k}$$

$$= 8\mathbf{i} - \mathbf{j} - 10\,\mathbf{k} \text{ at the point } (1, -2, -1).$$

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the vector $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$, then

$$\hat{\mathbf{a}}=\frac{2\,\mathbf{i}-\mathbf{j}-2\,\mathbf{k}}{\sqrt{(4+1+4)}}=\frac{2}{3}\,\mathbf{i}-\frac{1}{3}\,\mathbf{j}-\frac{2}{3}\,\mathbf{k}.$$

Therefore the required directional derivative is

$$\frac{df}{ds}=\text{grad } f\bullet\hat{\mathbf{a}}=(8\mathbf{i}-\mathbf{j}-10\mathbf{k})\bullet\left(\frac{2}{3}\,\mathbf{i}-\frac{1}{3}\,\mathbf{j}-\frac{2}{3}\,\mathbf{k}\right)$$

$$=\frac{16}{3}+\frac{1}{3}+\frac{20}{3}=\frac{37}{3}\cdot$$

Since this is positive, f is increasing in this direction.

Example 7: *Find the directional derivative of the function* $f=x^2-y^2+2z^2$ *at the point* $P\,(1,2,3)$ *in the direction of the line PQ where Q is the point* $(5,0,4)$.

(Purvanchal 2007; Kashi 14)

Solution: Here

$$\text{grad } f=\frac{\partial f}{\partial x}\,\mathbf{i}+\frac{\partial f}{\partial y}\,\mathbf{j}+\frac{\partial f}{\partial z}\,\mathbf{k}$$

$$=2x\mathbf{i}-2y\mathbf{j}+4z\mathbf{k}=2\mathbf{i}-4\mathbf{j}+12\mathbf{k}\text{ at the point }(1,2,3).$$

Also $\overrightarrow{PQ}$ = position vector of Q − position vector of P

$$=(5\mathbf{i}+0\,\mathbf{j}+4\mathbf{k})-(\mathbf{i}+2\,\mathbf{j}+3\mathbf{k})=4\mathbf{i}-2\mathbf{j}+\mathbf{k}\,.$$

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the vector $\overrightarrow{PQ}$, then

$$\hat{\mathbf{a}}=\frac{4\,\mathbf{i}-2\,\mathbf{j}+\mathbf{k}}{\sqrt{(16+4+1)}}=\frac{4\,\mathbf{i}-2\,\mathbf{j}+\mathbf{k}}{\sqrt{(21)}}\cdot$$

∴ the required directional derivative

$$=(\text{grad } f\,)\bullet\hat{\mathbf{a}}=(2\mathbf{i}-4\,\mathbf{j}+12\,\mathbf{k})\bullet\left\{\frac{4\mathbf{i}-2\,\mathbf{j}+\mathbf{k}}{\sqrt{(21)}}\right\}$$

$$=\frac{28}{\sqrt{(21)}}=\frac{28}{21}\sqrt{(21)}=\frac{4}{3}\sqrt{(21)}.$$

Example 8: *In what direction from the point* $(1,1,-1)$ *is the directional derivative of* $f=x^2-2y^2+4z^2$ *a maximum ? Also find the value of this maximum directional derivative.*

(Kanpur 2008)

Solution: We have

$$\text{grad } f=2x\,\mathbf{i}-4y\,\mathbf{j}+8z\,\mathbf{k}$$

$$=2\,\mathbf{i}-4\,\mathbf{j}-8\,\mathbf{k}\text{ at the point }(1,1,-1).$$

The directional derivative of f is maximum in the direction of grad f

$$=2\,\mathbf{i}-4\,\mathbf{j}-8\,\mathbf{k}\,.$$

The maximum value of this directional derivative

$$=|\text{grad } f\,|=|2\,\mathbf{i}-4\,\mathbf{j}-8\,\mathbf{k}|=\sqrt{(4+16+64)}=\sqrt{(84)}=2\sqrt{(21)}.$$

Example 9: *What is the greatest rate of increase of* $u = xyz^2$ *at the point* $(1, 0, 3)$?

Solution: We have $\nabla u = yz^2 \mathbf{i} + xz^2 \mathbf{j} + 2xyz \mathbf{k}$.

$\therefore$ at the point $(1, 0, 3)$, we have $\nabla u = 0 \mathbf{i} + 9 \mathbf{j} + 0 \mathbf{k} = 9 \mathbf{j}$.

The greatest rate of increase of u at the point $(1, 0, 3)$

$$= \text{the maximum value of } \frac{du}{ds} \text{ at the point } (1, 0, 3)$$

$$= |\nabla u|, \text{at the point } (1, 0, 3) \quad = |9\mathbf{j}| = 9.$$

Example 10: *Find the equations of the tangent plane and normal to the surface* $2xz^2 - 3xy - 4x = 7$ *at the point* $(1, -1, 2)$.

Solution: The equation of the surface is

$$f(x, y, z) \equiv 2xz^2 - 3xy - 4x = 7.$$

We have

$$\text{grad } f = (2z^2 - 3y - 4) \mathbf{i} - 3x \mathbf{j} + 4xz \mathbf{k}$$

$$= 7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}, \text{ at the point } (1, -1, 2).$$

$\therefore$ $7 \mathbf{i} - 3 \mathbf{j} + 8 \mathbf{k}$ is a vector along the normal to the surface at the point $(1, -1, 2)$.

The position vector of the point $(1, -1, 2)$ is $= \mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

If $\mathbf{R} = X \mathbf{i} + Y \mathbf{j} + Z \mathbf{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at $(1, -1, 2)$, then the vector $\mathbf{R} - \mathbf{r}$ is perpendicular to the vector grad f.

$\therefore$ the equation of the tangent plane is $(\mathbf{R} - \mathbf{r}) \bullet \text{grad } f = 0$

i.e., $\{(X \mathbf{i} + Y \mathbf{j} + Z \mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k})\} \bullet (7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}) = 0$

i.e., $\{(X - 1) \mathbf{i} + (Y + 1) \mathbf{j} + (Z - 2) \mathbf{k}\} \bullet (7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}) = 0$

i.e., $7(X - 1) - 3(Y + 1) + 8(Z - 2) = 0.$

The equations of the normal to the surface at the point $(1, -1, 2)$ are

$$\frac{X - 1}{\frac{\partial f}{\partial x}} = \frac{Y + 1}{\frac{\partial f}{\partial y}} = \frac{Z - 2}{\frac{\partial f}{\partial z}} \quad i.e., \quad \frac{X - 1}{7} = \frac{Y + 1}{-3} = \frac{Z - 2}{8}.$$

Example 11: *Find the equations of the tangent plane and normal to the surface* $xyz = 4$ *at the point* $(1, 2, 2)$. **(Meerut 2000)**

Solution: The equation of the surface is $f(x, y, z) \equiv x y z - 4 = 0$.

We have grad $f = yz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k} = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, at the point $(1, 2, 2)$.

$\therefore$ $4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ is a vector along the normal to the surface at the point $(1, 2, 2)$.

The position vector of the point $(1, 2, 2)$ is $= \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

If $\mathbf{R} = X \mathbf{i} + Y \mathbf{j} + Z \mathbf{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at $(1, 2, 2)$, the equation of the tangent plane is

$$(\mathbf{R} - \mathbf{r}) \bullet \text{grad } f = 0,$$

i.e., $\{(X\,\mathbf{i}+Y\,\mathbf{j}+Z\,\mathbf{k})-(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\}\bullet(4\mathbf{i}+2\mathbf{j}+2\mathbf{k})=0$

i.e., $\{(X-1)\,\mathbf{i}+(Y-2)\,\mathbf{j}+(Z-2)\,\mathbf{k}\}\bullet(4\mathbf{i}+2\mathbf{j}+2\mathbf{k})=0$

i.e., $4(X-1)+2(Y-2)+2(Z-2)=0$

i.e., $4X+2Y+2Z=12$, *i.e.,* $2X+Y+Z=6$.

The equations of the normal to the surface at the point (1, 2 , 2) are

$$\frac{X-1}{\dfrac{\partial f}{\partial x}}=\frac{Y-2}{\dfrac{\partial f}{\partial y}}=\frac{Z-2}{\dfrac{\partial f}{\partial z}}$$

i.e., $$\frac{X-1}{4}=\frac{Y-2}{2}=\frac{Z-2}{2},\quad i.e.,\quad \frac{X-1}{2}=\frac{Y-2}{1}=\frac{Z-2}{1}.$$

Example 12: *Find the angle between the surfaces* $x^2+y^2+z^2=9$, *and* $z=x^2+y^2-3$ *at the point* $(2,-1,2)$. **(Meerut 2001)**

Solution: Angle between two surfaces at a point is the angle between the normals to the surfaces at that point. Let $f_1=x^2+y^2+z^2$ and $f_2=x^2+y^2-z$.

Then grad $f_1=2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}$ and grad $f_2=2x\,\mathbf{i}+2y\,\mathbf{j}-\mathbf{k}$.

Let $\mathbf{n}_1=$ grad f_1 at the point $(2,-1,2)$ and $\mathbf{n}_2=$ grad f_2 at the point $(2,-1,2)$.

Then $\mathbf{n}_1=4\mathbf{i}-2\mathbf{j}+4\mathbf{k}$ and $\mathbf{n}_2=4\mathbf{i}-2\mathbf{j}-\mathbf{k}$.

The vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are along normals to the two surfaces at the point $(2,-1,2)$. If θ is the angle between these vectors, then

$$\mathbf{n}_1\bullet\mathbf{n}_2=|\mathbf{n}_1|\,|\mathbf{n}_2|\cos\theta$$

or $$16+4-4=\sqrt{(16+4+16)}\,\sqrt{(16+4+1)}\cos\theta.$$

$\therefore$ $$\cos\theta=\frac{16}{6\sqrt{(21)}}\quad\text{or}\quad\theta=\cos^{-1}\frac{8}{3\sqrt{(21)}}.$$

Comprehensive Exercise 2

1. (i) Find the gradient and the unit normal to the level surface $x^2+y-z=4$ at the point $(2,0,0)$.

(ii) Find the unit normal to the surface $z=x^2+y^2$ at the point $(-1,-2,5)$.

2. (i) Find the unit vector normal to the surface $x^2-y^2+z=2$ at the point $(1,-1,2)$.

(ii) Find the unit normal to the surface $x^4-3xyz+z^2+1=0$ at the point $(1,1,1)$.

(iii) Find a unit normal vector to the surface $x^2y+2xz=4$ at the point $(2,-2,3)$.

3. (i) Find the directional derivatives of a scalar point function f in the direction of coordinate axes.

(ii) Find the directional derivative of $\phi = xy + yz + zx$ in the direction of the vector $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ at $(1, 2, 0)$. **(Kumaun 2015)**

(iii) Find the directional derivative of $\phi(x, y, z) = x^2 yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

(iv) Find the directional derivative of $f(x, y, z) = x^2 - 2y^2 + 4z^2$ at the point $(1, 1, -1)$ in the direction of $2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

(v) Find the directional derivative of the function $f = xy + yz + zx$ in the direction of the vector $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ at the point $(3, 1, 2)$.

(vi) Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t, y = t^2, z = t^3$ at the point $(1, 1, 1)$.

4. Find the directional derivatives of $\phi = xyz$ at the point $(2, 2, 2)$, in the directions (i) $\mathbf{i}$, (ii) $\mathbf{j}$, (iii) $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

5. For the function $f = y/(x^2 + y^2)$, find the value of the directional derivative making an angle 30° with the positive x-axis at the point $(0, 1)$.

6. (i) Find the greatest value of the directional derivative of the function $2x^2 - y - z^4$ at the point $(2, -1, 1)$.

(ii) In what direction the directional derivative of $\phi = x^2 y^2 z$ from $(1, 1, 2)$ will be maximum and what is its magnitude ? Also find a unit normal vector to the surface $x^2 y^2 z = 2$ at the point $(1, 1, 2)$.

(iii) Find the maximum value of the directional derivative of $\phi = x^2 yz$ at the point $(1, 4, 1)$.

(iv) Calculate the maximum rate of change and the corresponding direction for the function $\phi = x^2 y^3 z^4$ at the point $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

(v) Find the values of the constants a, b, c so that the directional derivative of $\phi = ax^2 + by^2 + cz^2$ at $(1, 1, 2)$ has a maximum magnitude 4 in the direction parallel to y-axis.

7. Find the equation of the tangent plane to the surface $yz - zx + xy + 5 = 0$, at the point $(1, -1, 2)$.

8. (i) Find the equations of the tangent plane and normal to the surface $x^2 + y^2 + z^2 = 25$ at the point $(4, 0, 3)$.

(ii) Given the curve $x^2 + y^2 + z^2 = 1, x + y + z = 1$ (intersection of two surfaces), find the equations of the tangent line at the point $(1, 0, 0)$.

(iii) Find the equations of the tangent plane and normal to the surface $z = x^2 + y^2$ at the point $(2, -1, 5)$.

9. (i) Find the equations of the tangent plane and the normal to the surface $x^2 + 2y^2 + 3z^2 = 12$ at the point $(1, 2, -1)$.

(ii) Find the equations of the tangent plane and the normal to the surface $xy + yz + zx = 1$, at the point $(2, 3, -1)$.

(iii) Find the equations of the tangent plane and the normal to the surface $z = x^2 - 2xy - y^2$ at the point $(1, 2, -7)$.

(iv) Find the equation of the tangent plane to the surface $x^2 + y^2 + z^2 = 9$ at $(2, -1, 2)$.

(v) Find the equation of the tangent plane to the surface $z = x^2 + y^2$ at the point $(1, -1, 2)$.

10. Show that the directional derivative of a scalar point function at any point along any tangent line to the level surface at the point is zero.

11. If $\mathbf{F}$ and f are point functions, show that the components of the former, tangential and normal to the level surface $f = 0$ are

$$\frac{\nabla f \times (\mathbf{F} \times \nabla f)}{(\nabla f)^2} \text{ and } \frac{(\mathbf{F} \bullet \nabla f)\, \nabla f}{(\nabla f)^2}.$$

12. Find the angle of intersection at $(4, -3, 2)$ of spheres $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 = 0$.

13. Find the constants a and b so that the surface $ax^2 - byz = (a + 2)x$ will be orthogonal to the surface $4x^2 y + z^3 = 4$ at the point $(1, -1, 2)$.

14. Show that the sum of the squares of the intercepts on the coordinate axes made by the tangent plane to the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ is constant.

Answers 2

1. (i) $4\mathbf{i} + \mathbf{j} - \mathbf{k}$; $\dfrac{1}{3\sqrt{2}}(4\mathbf{i} + \mathbf{j} - \mathbf{k})$ (ii) $\dfrac{-(2\mathbf{i} + 4\mathbf{j} + \mathbf{k})}{\sqrt{(21)}}$

2. (i) $\dfrac{1}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ (ii) $\dfrac{\mathbf{i} - 3\mathbf{j} - \mathbf{k}}{\sqrt{(11)}}$ (iii) $\left(-\dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right)$

3. (i) $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$ in the directions of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$

(ii) $\dfrac{10}{3}$ (iii) $-\dfrac{13}{3}$ (iv) $\dfrac{8}{\sqrt{6}}$

(v) $\dfrac{45}{7}$ (vi) $\dfrac{18}{\sqrt{(14)}}$

4. (i) 4 (ii) 4 (iii) $4\sqrt{3}$

5. $-1/2$

6. (i) 9

(ii) $\sqrt{(33)}$ in the direction of the vector $4\,\mathbf{i}+4\,\mathbf{j}+\mathbf{k}$; $\dfrac{4\,\mathbf{i}+4\,\mathbf{j}+\mathbf{k}}{\sqrt{(33)}}$

(iii) 9

(iv) $324\sqrt{2}$ in the direction of the vector $108\,(\mathbf{i}+\mathbf{j}-4\,\mathbf{k})$

(v) $a=0, b=2, c=0$

7. $3X-3Y+2Z=10$

8. (i) $4x+3z=25$; $\dfrac{x-4}{4}=\dfrac{y}{0}=\dfrac{z-3}{3}$

(ii) $X=1, \dfrac{Y}{-1}=\dfrac{Z}{1}$ (iii) $4x-2y-z=5$; $\dfrac{x-2}{4}=\dfrac{y+1}{-2}=\dfrac{z-5}{-1}$

9. (i) $x+4y-3z=12$; $\dfrac{x-1}{1}=\dfrac{y-2}{4}=\dfrac{z+1}{-3}$

(ii) $2x+y+5z=2$; $\dfrac{x-2}{2}=\dfrac{y-3}{1}=\dfrac{z+1}{5}$

(iii) $2x+6y+z=7$; $\dfrac{x-1}{2}=\dfrac{y-2}{6}=\dfrac{z+7}{1}$

(iv) $2x-y+2z=9$ (v) $2x-2y-z=2$

12. $\cos^{-1}\sqrt{(19/29)}$ **13.** $a=5/2, b=1$

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8 Divergence of a Vector Point Function

(Agra 2005)

Definition: *Let* **V** *be any given differentiable vector point function. Then the divergence of* **V**, *written as,* $\nabla \bullet \mathbf{V}$ or *div* **V**,

is defined as
$$\text{div}\,\mathbf{V}=\nabla\bullet\mathbf{V}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\bullet\mathbf{V}$$
$$=\mathbf{i}\bullet\frac{\partial\mathbf{V}}{\partial x}+\mathbf{j}\bullet\frac{\partial\mathbf{V}}{\partial y}+\mathbf{k}\bullet\frac{\partial\mathbf{V}}{\partial z}=\Sigma\,\mathbf{i}\bullet\frac{\partial\mathbf{V}}{\partial x}\cdot$$

It should be noted that div **V** is a scalar quantity. Thus *the divergence of a vector point function is a scalar point function.*

Theorem: *If* $\mathbf{V}=V_1\,\mathbf{i}+V_2\,\mathbf{j}+V_3\,\mathbf{k}$ *is a differentiable vector point function, then*
$$\text{div}\,\mathbf{V}=\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}\cdot$$

Proof: We have by definition

$$\text{div } \mathbf{V} = \nabla \bullet \mathbf{V} = \mathbf{i} \bullet \frac{\partial \mathbf{V}}{\partial x} + \mathbf{j} \bullet \frac{\partial \mathbf{V}}{\partial y} + \mathbf{k} \bullet \frac{\partial \mathbf{V}}{\partial z} \cdot$$

Now $\mathbf{V} = V_1 \mathbf{i} + V_2 \mathbf{j} + V_3 \mathbf{k}$.

$\therefore$ $$\frac{\partial \mathbf{V}}{\partial x} = \frac{\partial V_1}{\partial x} \mathbf{i} + \frac{\partial V_2}{\partial x} \mathbf{j} + \frac{\partial V_3}{\partial x} \mathbf{k}.$$

$\therefore$ $$\mathbf{i} \bullet \frac{\partial \mathbf{V}}{\partial x} = \mathbf{i} \bullet \left(\frac{\partial V_1}{\partial x} \mathbf{i} + \frac{\partial V_2}{\partial x} \mathbf{j} + \frac{\partial V_3}{\partial x} \mathbf{k} \right) = \frac{\partial V_1}{\partial x} \cdot$$

Similarly, $$\mathbf{j} \bullet \frac{\partial \mathbf{V}}{\partial y} = \frac{\partial V_2}{\partial y} \quad \text{and} \quad \mathbf{k} \bullet \frac{\partial \mathbf{V}}{\partial z} = \frac{\partial V_3}{\partial z} \cdot$$

Hence, $$\text{div } \mathbf{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \cdot$$

Solenoidal Vector: *A vector* $\mathbf{V}$ *is said to be solenoidal if div* $\mathbf{V} = 0$.

9 Curl of a Vector Point Function

(Agra 2005)

Definition: *Let* $\mathbf{f}$ *be any given differentiable vector point function. Then the curl or rotation of* $\mathbf{f}$ *, written as* $\nabla \times \mathbf{f}$ *, curl* $\mathbf{f}$ *or rot* $\mathbf{f}$ *is defined as*

$$\textit{curl } \mathbf{f} = \nabla \times \mathbf{f} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times \mathbf{f}$$

$$= \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{f}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{f}}{\partial z} = \sum \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} \cdot$$

It should be noted that curl $\mathbf{f}$ is a vector quantity. *Thus the curl of a vector point function is a vector point function.*

Theorem: *If* $\mathbf{f} = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}$ *is a differentiable vector point function, then*

$$\textit{curl } \mathbf{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathbf{k} \ .$$

Proof: We have by definition

$$\text{curl } \mathbf{f} = \nabla \times \mathbf{f} = \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{f}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{f}}{\partial z}$$

$$= \mathbf{i} \times \frac{\partial}{\partial x} (f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}) + \mathbf{j} \times \frac{\partial}{\partial y} (f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k})$$

$$+ \mathbf{k} \times \frac{\partial}{\partial z} (f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k})$$

$$= \mathbf{i} \times \left(\frac{\partial f_1}{\partial x} \mathbf{i} + \frac{\partial f_2}{\partial x} \mathbf{j} + \frac{\partial f_3}{\partial x} \mathbf{k} \right) + \mathbf{j} \times \left(\frac{\partial f_1}{\partial y} \mathbf{i} + \frac{\partial f_2}{\partial y} \mathbf{j} + \frac{\partial f_3}{\partial y} \mathbf{k} \right)$$

$$+ \mathbf{k} \times \left(\frac{\partial f_1}{\partial z} \mathbf{i} + \frac{\partial f_2}{\partial z} \mathbf{j} + \frac{\partial f_3}{\partial z} \mathbf{k} \right)$$

$$= \left(\frac{\partial f_2}{\partial x}\,\mathbf{k} - \frac{\partial f_3}{\partial x}\,\mathbf{j}\right) + \left(-\frac{\partial f_1}{\partial y}\,\mathbf{k} + \frac{\partial f_3}{\partial y}\,\mathbf{i}\right) + \left(\frac{\partial f_1}{\partial z}\,\mathbf{j} - \frac{\partial f_2}{\partial z}\,\mathbf{i}\right)$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}.$$

Note: It should be noted that the expression for curl **f** can be written immediately if we treat the operator ∇ as a vector quantity. Thus

$$\text{Curl } \mathbf{f} = \nabla \times \mathbf{f} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_2 & f_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ f_1 & f_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ f_1 & f_2 \end{vmatrix}\mathbf{k}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}.$$

But we must take care that in the expansion of the determinant the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ must *precede* the functions f_1, f_2, f_3.

Irrotational vector: *A vector* **f** *is said to be irrotational if* $\nabla \times \mathbf{f} = \mathbf{0}$.

10 The Laplacian Operator ∇^2

The Laplacian operator ∇^2 *is defined as* $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

If f is a scalar point function, then $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

It should be noted that $\nabla^2 f$ is also a scalar quantity.

If **f** is a vector point function, then $\nabla^2 \mathbf{f} = \frac{\partial^2 \mathbf{f}}{\partial x^2} + \frac{\partial^2 \mathbf{f}}{\partial y^2} + \frac{\partial^2 \mathbf{f}}{\partial z^2}$.

It should be noted that $\nabla^2 \mathbf{f}$ is also a vector quantity.

Laplace's equation: *The equation* $\nabla^2 f = 0$ *is called Laplace's equation.* A function which satisfies Laplace's equation is called a **harmonic function.**

Illustrative Examples

Example 13: *Prove that div* $\mathbf{r} = 3$. **(Kumaun 2000; Garhwal 01; Bundelkhand 01, 04; Meerut 03, 04, 07B, 08; Agra 08; Lucknow 05; Kanpur 13; Avadh 09; Purvanchal 11)**

Solution: We have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

By definition, $$\text{div}\,\mathbf{r} = \nabla \bullet \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet \mathbf{r} = \mathbf{i} \bullet \frac{\partial \mathbf{r}}{\partial x} + \mathbf{j} \bullet \frac{\partial \mathbf{r}}{\partial y} + \mathbf{k} \bullet \frac{\partial \mathbf{r}}{\partial z}$$

$$= \mathbf{i} \bullet \mathbf{i} + \mathbf{j} \bullet \mathbf{j} + \mathbf{k} \bullet \mathbf{k} \qquad \left[\because \frac{\partial \mathbf{r}}{\partial x} = \mathbf{i}, \frac{\partial \mathbf{r}}{\partial y} = \mathbf{j}, \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}\right]$$

$$= 1 + 1 + 1 = 3.$$

Example 14: *Prove that curl* $\mathbf{r} = \mathbf{0}$.

(Kumaun 2000; Garhwal 02; Meerut 11; Bundelkhand 06; Agra 08; Avadh 09; Kashi 13; Purvanchal 11; Rohilkhand 14)

Solution: We have by definition

$$\text{curl}\,\mathbf{r} = \nabla \times \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times \mathbf{r}$$

$$= \mathbf{i} \times \frac{\partial \mathbf{r}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{r}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{r}}{\partial z}.$$

Now $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

$\therefore$ $\dfrac{\partial \mathbf{r}}{\partial x} = \mathbf{i}, \dfrac{\partial \mathbf{r}}{\partial y} = \mathbf{j}, \dfrac{\partial \mathbf{r}}{\partial z} = \mathbf{k}$.

$\therefore$ $$\text{curl}\,\mathbf{r} = \mathbf{i} \times \mathbf{i} + \mathbf{j} \times \mathbf{j} + \mathbf{k} \times \mathbf{k}$$
$$= \mathbf{0} + \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Example 15: *If* $\mathbf{f} = x^2 y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, *find*

(i) *div* $\mathbf{f}$, **(Kumaun 2008)**

(ii) *curl* $\mathbf{f}$, **(Meerut 2012)**

(iii) *curl curl* $\mathbf{f}$. **(Meerut 2001; Kanpur 14)**

Solution: (i) We have

$$\text{div}\,\mathbf{f} = \nabla \bullet \mathbf{f} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet (x^2 y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k})$$

$$= \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(2yz)$$

$$= 2xy + 0 + 2y$$

$$= 2y(x + 1).$$

(ii) We have curl $\mathbf{f} = \nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & -2xz & 2yz \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2 y)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2 y)\right]\mathbf{k}$$

$$= (2z + 2x)\,\mathbf{i} - 0\,\mathbf{j} + (-2z - x^2)\,\mathbf{k}$$

$$= (2x + 2z)\,\mathbf{i} - (x^2 + 2z)\,\mathbf{k}\,.$$

(iii) We have curl curl $\mathbf{f} = \nabla \times (\nabla \times \mathbf{f}) = \nabla \times [(2x + 2z)\,\mathbf{i} - (x^2 + 2z)\,\mathbf{k}]$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x + 2z & 0 & -x^2 - 2z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-x^2 - 2z)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-x^2 - 2z) - \frac{\partial}{\partial z}(2x + 2z)\right]\mathbf{j}$$

$$+ \left[0 - \frac{\partial}{\partial y}(2x + 2z)\right]\mathbf{k}$$

$$= 0\,\mathbf{i} - (-2x - 2)\,\mathbf{j} + (0 - 0)\,\mathbf{k} = (2x + 2)\,\mathbf{j}.$$

Example 16: *Determine the constant a so that the vector*
$\mathbf{V} = (x + 3y)\,\mathbf{i} + (y - 2z)\,\mathbf{j} + (x + az)\,\mathbf{k}$ *is solenoidal.*
(Rohilkhand 2009B)

Solution: A vector **V** is said to be solenoidal if div **V** = 0.

We have div $\mathbf{V} = \nabla \bullet \mathbf{V} = \dfrac{\partial}{\partial x}(x + 3y) + \dfrac{\partial}{\partial y}(y - 2z) + \dfrac{\partial}{\partial z}(x + az)$

$$= 1 + 1 + a = 2 + a.$$

Now div $\mathbf{V} = 0$ if $2 + a = 0$ *i.e.*, if $a = -2$.

Example 17: *Show that the vector* $\mathbf{V} = (\sin y + z)\,\mathbf{i} + (x\cos y - z)\,\mathbf{j} + (x - y)\,\mathbf{k}$ *is irrotational.*

Solution: A vector **V** is said to be irrotational if curl **V** = **0.** We have

$$\text{curl } \mathbf{V} = \nabla \times \mathbf{V}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x - y \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(x - y) - \frac{\partial}{\partial z}(x \cos y - z)\right] \mathbf{i} - \left[\frac{\partial}{\partial x}(x - y) - \frac{\partial}{\partial z}(\sin y + z)\right] \mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(x \cos y - z) - \frac{\partial}{\partial y}(\sin y + z)\right] \mathbf{k}$$

$$= (-1 + 1)\,\mathbf{i} - (1 - 1)\,\mathbf{j} + (\cos y - \cos y)\,\mathbf{k} = \mathbf{0}.$$

$\therefore$ $\mathbf{V}$ is irrotational.

Example 18: *Prove that* $\nabla \bullet (r^3 \mathbf{r}) = 6r^3$. **(Purvanchal 2006, 10)**

Solution: We have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

$\therefore$ $r^3 \mathbf{r} = r^3 (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = r^3 x\,\mathbf{i} + r^3 y\,\mathbf{j} + r^3 z\,\mathbf{k}$.

$\therefore$ $\nabla \bullet (r^3 \mathbf{r}) = \text{div}\,(r^3 \mathbf{r})$

$$= \frac{\partial}{\partial x}(r^3 x) + \frac{\partial}{\partial y}(r^3 y) + \frac{\partial}{\partial z}(r^3 z)$$

$$= r^3 + 3r^2 x \frac{\partial r}{\partial x} + r^3 + 3r^2 y \frac{\partial r}{\partial y} + r^3 + 3r^2 z \frac{\partial r}{\partial z}$$

$$= 3r^3 + 3r^2 \left(x \frac{\partial r}{\partial x} + y \frac{\partial r}{\partial y} + z \frac{\partial r}{\partial z}\right) \qquad \ldots(1)$$

Now $r^2 = x^2 + y^2 + z^2$.

$\therefore$ $2r \dfrac{\partial r}{\partial x} = 2x$ or $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}\cdot$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}\cdot$

$\therefore$ from (1), $\nabla \bullet (r^3 \mathbf{r}) = 3r^3 + 3r^2 \left(x \cdot \dfrac{x}{r} + y \cdot \dfrac{y}{r} + z \cdot \dfrac{z}{r}\right)$

$$= 3r^3 + 3r^2 \left(\frac{x^2 + y^2 + z^2}{r}\right)$$

$$= 3r^3 + 3r^2 \cdot \frac{r^2}{r} = 3r^3 + 3r^3 = 6r^3.$$

Example 19: *If* $\mathbf{V}$ *is a constant vector, show that*

(i) *div* $\mathbf{V} = 0$, **(Rohilkhand 2005; Purvanchal 14)**

(ii) *curl* $\mathbf{V} = \mathbf{0}$. **(Purvanchal 2014; Kumaun 15)**

Solution: (i) We have

$$\text{div}\,\mathbf{V} = \mathbf{i} \bullet \frac{\partial \mathbf{V}}{\partial x} + \mathbf{j} \bullet \frac{\partial \mathbf{V}}{\partial y} + \mathbf{k} \bullet \frac{\partial \mathbf{V}}{\partial z}$$

$$= \mathbf{i} \bullet \mathbf{0} + \mathbf{j} \bullet \mathbf{0} + \mathbf{k} \bullet \mathbf{0} = 0.$$

(ii) We have

$$\text{curl } \mathbf{V} = \mathbf{i} \times \frac{\partial \mathbf{V}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{V}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{V}}{\partial z}$$

$$= \mathbf{i} \times \mathbf{0} + \mathbf{j} \times \mathbf{0} + \mathbf{k} \times \mathbf{0} = \mathbf{0}.$$

Example 20: *If* **a** *is a constant vector, find*

(i) *div* **(r × a)** , **(Bundelkhand 2006; Kumaun 07, 11)**

(ii) *curl* **(r × a)**. **(Bundelkhand 2006; Kanpur 06; Kumaun 07; Agra 08; Rohilkhand 14)**

Solution: We have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

Let $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$. Then the scalars a_1, a_2, a_3 are all constants.

We have

$$\mathbf{r} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= (a_3 y - a_2 z)\,\mathbf{i} + (a_1 z - a_3 x)\,\mathbf{j} + (a_2 x - a_1 y)\,\mathbf{k}.$$

(i) div **(r × a)** $= \dfrac{\partial}{\partial x}(a_3 y - a_2 z) + \dfrac{\partial}{\partial y}(a_1 z - a_3 x) + \dfrac{\partial}{\partial z}(a_3 x - a_1 y)$

$$= 0 + 0 + 0 = 0.$$

(ii) curl **(r × a)** $= \nabla \times (\mathbf{r} \times \mathbf{a})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_3 y - a_2 z & a_1 z - a_3 x & a_2 x - a_1 y \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(a_2 x - a_1 y) - \frac{\partial}{\partial z}(a_1 z - a_3 x)\right]\mathbf{i}$$

$$- \left[\frac{\partial}{\partial x}(a_2 x - a_1 y) - \frac{\partial}{\partial z}(a_3 y - a_2 z)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(a_1 z - a_3 x) - \frac{\partial}{\partial y}(a_3 y - a_2 z)\right]\mathbf{k}$$

$$= -2a_1\,\mathbf{i} - 2a_2\,\mathbf{j} - 2a_3\,\mathbf{k} = -2\,(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) = -2\mathbf{a}.$$

Example 21: *If* $\mathbf{V} = e^{xyz}\,(\mathbf{i} + \mathbf{j} + \mathbf{k})$, *find curl* **V.**

Solution: We have curl $\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{xyz} & e^{xyz} & e^{xyz} \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(e^{xyz}) - \frac{\partial}{\partial z}(e^{xyz})\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(e^{xyz}) - \frac{\partial}{\partial z}(e^{xyz})\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(e^{xyz}) - \frac{\partial}{\partial y}(e^{xyz})\right]\mathbf{k}$$

$$= e^{xyz}\,(xz - xy)\,\mathbf{i} + e^{xyz}\,(xy - yz)\,\mathbf{j} + e^{xyz}\,(yz - xz)\,\mathbf{k}.$$

Example 22: *Evaluate div* $\mathbf{f}$ *where* $\mathbf{f} = 2x^2 z\,\mathbf{i} - xy^2 z\,\mathbf{j} + 3y^2 x\,\mathbf{k}$.

Solution: We have

$$\text{div}\,\mathbf{f} = \nabla \bullet \mathbf{f} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet (2x^2 z\,\mathbf{i} - xy^2 z\,\mathbf{j} + 3y^2 x\,\mathbf{k})$$

$$= \frac{\partial}{\partial x}(2x^2 z) + \frac{\partial}{\partial y}(-xy^2 z) + \frac{\partial}{\partial z}(3y^2 x)$$

$$= 4xz - 2xyz + 0 = 2xz\,(2 - y).$$

Example 23: *Show that* $\nabla^2\,(x / r^3) = 0$. **(Kumaun 2008)**

Solution: $$\nabla^2\left(\frac{x}{r^3}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{r^3}\right).$$

Now $$\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)\right\} = \frac{\partial}{\partial x}\left\{\frac{1}{r^3} - \frac{3x}{r^4}\frac{\partial r}{\partial x}\right\}$$

$$= \frac{\partial}{\partial x}\left\{\frac{1}{r^3} - \frac{3x}{r^4}\frac{x}{r}\right\} \quad \left[\because\ r^2 = x^2 + y^2 + z^2 \text{ gives } \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

$$= \frac{\partial}{\partial x}\left\{\frac{1}{r^3} - \frac{3x^2}{r^5}\right\} = -\frac{3}{r^4}\frac{\partial r}{\partial x} - \frac{6x}{r^5} + \frac{15x^2}{r^6}\frac{\partial r}{\partial x}$$

$$= -\frac{3}{r^4}\frac{x}{r} - \frac{6x}{r^5} + \frac{15x^2}{r^6}\frac{x}{r} = -\frac{9x}{r^5} + \frac{15x^3}{r^7}.$$

Again $$\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right\} = \frac{\partial}{\partial y}\left\{-\frac{3x}{r^4}\frac{\partial r}{\partial y}\right\}$$

$$= \frac{\partial}{\partial y}\left\{-\frac{3x}{r^4}\frac{y}{r}\right\} \quad \left[\because \frac{\partial r}{\partial y} = \frac{y}{r}\right]$$

$$= \frac{\partial}{\partial y}\left(-\frac{3xy}{r^5}\right) = -\frac{3x}{r^5} + \frac{15xy}{r^6}\frac{\partial r}{\partial y} = -\frac{3x}{r^5} + \frac{15x\,y^2}{r^7}.$$

Similarly $$\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right) = -\frac{3x}{r^5} + \frac{15xz^2}{r^7}.$$

Therefore, adding we get

$$\nabla^2\left(\frac{x}{r^3}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{r^3}\right)$$

$$= -\frac{9x}{r^5} + \frac{15x^3}{r^7} - \frac{3x}{r^5} + \frac{15xy^2}{r^7} - \frac{3x}{r^5} + \frac{15xz^2}{r^7}$$

$$= -\frac{15x}{r^5} + \frac{15x}{r^7}(x^2 + y^2 + z^2) = -\frac{15x}{r^5} + \frac{15x}{r^7} r^2 = 0.$$

Comprehensive Exercise 3

1. (i) If $\mathbf{F} = x^2 z \, \mathbf{i} - 2y^3 z^2 \, \mathbf{j} + xy^2 z \, \mathbf{k}$, find div $\mathbf{F}$, curl $\mathbf{F}$ at $(1, -1, 1)$.

(ii) If $\mathbf{f} = (y^2 + z^2 - x^2)\,\mathbf{i} + (z^2 + x^2 - y^2)\,\mathbf{j} + (x^2 + y^2 - z^2)\,\mathbf{k}$, find div $\mathbf{f}$ and curl $\mathbf{f}$.

(iii) If $\mathbf{f} = xy^2 \, \mathbf{i} + 2x^2 yz \, \mathbf{j} - 3yz^2 \, \mathbf{k}$, find div $\mathbf{f}$ and curl $\mathbf{f}$. What are their values at the point $(1, -1, 1)$? **(Rohilkhand 2005)**

2. Find div $\mathbf{F}$ and curl $\mathbf{F}$ where $\mathbf{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$. **(Rohilkhand 2007)**

3. Find the divergence and curl of the vector
$\mathbf{f} = (x^2 - y^2)\,\mathbf{i} + 2xy\mathbf{j} + (y^2 - xy)\,\mathbf{k}$. **(Bundelkhand 2004)**

4. Given $\phi = 2x^3 y^2 z^4$, find div (grad ϕ).

5. If $u = x^2 - y^2 + 4z$, show that $\nabla^2 u = 0$.

6. If $u = 3x^2 y$ and $v = xz^2 - 2y$, then find grad [(grad u) • (grad v)].

7. If $\mathbf{f} = (x + y + 1)\,\mathbf{i} + \mathbf{j} + (-x - y)\,\mathbf{k}$, prove that $\mathbf{f} \bullet \text{curl}\,\mathbf{f} = 0$.

8. (i) If $\mathbf{f} = f_1 \, \mathbf{i} + f_2 \, \mathbf{j} + f_3 \, \mathbf{k}$, show that
$\nabla \bullet \mathbf{f} = \nabla f_1 \bullet \mathbf{i} + \nabla f_2 \bullet \mathbf{j} + \nabla f_3 \bullet \mathbf{k}$. **(Bundelkhand 2001)**

(ii) Prove that $\nabla \bullet (r^3 \, \mathbf{r}) = 6\, r^3$.

9. Find the constants a, b, c so that the vector
$\mathbf{F} = (x + 2y + az)\,\mathbf{i} + (bx - 3y - z)\,\mathbf{j} + (4x + cy + 2z)\,\mathbf{k}$ is irrotational.
(Bundelkhand 2005; Rohilkhand 08B)

10. Show that the vector $\mathbf{F} = 3y^4 z^2 \, \mathbf{i} + 4x^3 z^2 \, \mathbf{j} - 3x^2 y^2 \, \mathbf{k}$ is solenoid but not irrotational. **(Kashi 2014)**

Answers 3

1. (i) $-3; -6i$

(ii) $-2(x + y + z)$; $2(y - z)\,\mathbf{i} + 2(z - x)\,\mathbf{j} + 2(x - y)\,\mathbf{k}$

(iii) $y^2 + 2x^2 z - 6yz; 9$; $-(3z^2 + 2x^2 y)\,\mathbf{i} + (4xyz - 2xy)\,\mathbf{k}$; $-\mathbf{i} - 2\mathbf{k}$

2. $6(x + y + z); 0$ 3. $4x; (2y - x)\,\mathbf{i} + y\,\mathbf{j} + 4y\,\mathbf{k}$

4. $12xy^2z^4 + 4x^3z^4 + 24\,x^3y^2z^2$

6. $(6yz^2 - 12x)\,\mathbf{i} + (6xz^2)\,\mathbf{j} + (12xyz)\,\mathbf{k}$

9. $a = 4, b = 2, c = -1$

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11 Vector Identities

1. *Prove that div* **(A + B)** = *div* **A** + *div* **B** *or* $\nabla \bullet (\mathbf{A} + \mathbf{B}) = \nabla \bullet \mathbf{A} + \nabla \bullet \mathbf{B}$.

Proof: We have

$$\text{div }(\mathbf{A} + \mathbf{B}) = \nabla \bullet (\mathbf{A} + \mathbf{B}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet (\mathbf{A} + \mathbf{B})$$

$$= \mathbf{i} \bullet \frac{\partial}{\partial x}(\mathbf{A} + \mathbf{B}) + \mathbf{j} \bullet \frac{\partial}{\partial y}(\mathbf{A} + \mathbf{B}) + \mathbf{k} \bullet \frac{\partial}{\partial z}(\mathbf{A} + \mathbf{B})$$

$$= \mathbf{i} \bullet \left(\frac{\partial \mathbf{A}}{\partial x} + \frac{\partial \mathbf{B}}{\partial x}\right) + \mathbf{j} \bullet \left(\frac{\partial \mathbf{A}}{\partial y} + \frac{\partial \mathbf{B}}{\partial y}\right) + \mathbf{k} \bullet \left(\frac{\partial \mathbf{A}}{\partial z} + \frac{\partial \mathbf{B}}{\partial z}\right)$$

$$= \left(\mathbf{i} \bullet \frac{\partial \mathbf{A}}{\partial x} + \mathbf{j} \bullet \frac{\partial \mathbf{A}}{\partial y} + \mathbf{k} \bullet \frac{\partial \mathbf{A}}{\partial z}\right) + \left(\mathbf{i} \bullet \frac{\partial \mathbf{B}}{\partial x} + \mathbf{j} \bullet \frac{\partial \mathbf{B}}{\partial y} + \mathbf{k} \bullet \frac{\partial \mathbf{B}}{\partial z}\right)$$

$$= \nabla \bullet \mathbf{A} + \nabla \bullet \mathbf{B} = \text{div }\mathbf{A} + \text{div }\mathbf{B}.$$

2. *Prove that curl* **(A + B)** = *curl* **A** + *curl* **B** *or* $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$.

(Rohilkhand 2008)

Proof: We have curl $(\mathbf{A} + \mathbf{B}) = \nabla \times (\mathbf{A} + \mathbf{B})$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (\mathbf{A} + \mathbf{B})$$

$$= \Sigma\, \mathbf{i} \times \frac{\partial}{\partial x}(\mathbf{A} + \mathbf{B}) = \Sigma\, \mathbf{i} \times \left(\frac{\partial \mathbf{A}}{\partial x} + \frac{\partial \mathbf{B}}{\partial x}\right)$$

$$= \Sigma\, \mathbf{i} \times \frac{\partial \mathbf{A}}{\partial x} + \Sigma\, \mathbf{i} \times \frac{\partial \mathbf{B}}{\partial x} = \text{curl }\mathbf{A} + \text{curl }\mathbf{B}.$$

3. *If* **A** *is a differentiable vector function and* ϕ *is a differentiable scalar function, then*

div $(\phi\mathbf{A}) = (\textit{grad}\ \phi) \bullet \mathbf{A} + \phi\ \textit{div}\ \mathbf{A}$ *or* $\nabla \bullet (\phi\mathbf{A}) = (\nabla\phi) \bullet \mathbf{A} + \phi\,(\nabla \bullet \mathbf{A})$.

(Garhwal 2001; Agra 05; Kashi 14)

Proof: We have

$$\text{div }(\phi\mathbf{A}) = \nabla \bullet (\phi\mathbf{A}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet (\phi\mathbf{A})$$

$$= \mathbf{i} \bullet \frac{\partial}{\partial x}(\phi\mathbf{A}) + \mathbf{j} \bullet \frac{\partial}{\partial y}(\phi\mathbf{A}) + \mathbf{k} \bullet \frac{\partial}{\partial z}(\phi\mathbf{A})$$

$$= \Sigma\left\{\mathbf{i} \bullet \left(\frac{\partial}{\partial x}(\phi\mathbf{A})\right)\right\} = \Sigma\left\{\mathbf{i} \bullet \left(\frac{\partial \phi}{\partial x}\mathbf{A} + \phi\frac{\partial \mathbf{A}}{\partial x}\right)\right\}$$

$$= \Sigma\left\{\mathbf{i} \bullet \left(\frac{\partial \phi}{\partial x}\mathbf{A}\right)\right\} + \Sigma\left\{\mathbf{i} \bullet \left(\phi\frac{\partial \mathbf{A}}{\partial x}\right)\right\}$$

$$= \Sigma\left\{\left(\frac{\partial \phi}{\partial x}\mathbf{i}\right) \bullet \mathbf{A}\right\} + \Sigma\left\{\phi\left(\mathbf{i} \bullet \frac{\partial \mathbf{A}}{\partial x}\right)\right\}$$

$$[\because\ \mathbf{a} \bullet (m\mathbf{b}) = (m\mathbf{a}) \bullet \mathbf{b} = m(\mathbf{a} \bullet \mathbf{b})]$$

$$= \left\{\Sigma\frac{\partial \phi}{\partial x}\mathbf{i}\right\} \bullet \mathbf{A} + \phi\,\Sigma\left(\mathbf{i} \bullet \frac{\partial \mathbf{A}}{\partial x}\right) = (\nabla\phi) \bullet \mathbf{A} + \phi(\nabla \bullet \mathbf{A}).$$

4. *Prove that curl* ($\phi\mathbf{A}$) = (*grad* ϕ) $\times$ **A** + ϕ *curl* **A**

or $\nabla \times (\phi\mathbf{A}) = (\nabla\phi) \times \mathbf{A} + \phi(\nabla \times \mathbf{A})$.

(Garhwal 2002, 03; Bundelkhand 06)

Proof: We have curl (ϕ**A**) $= \nabla \times (\phi\mathbf{A}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (\phi\mathbf{A})$

$$= \Sigma\left\{\mathbf{i} \times \frac{\partial}{\partial x}(\phi\mathbf{A})\right\} = \Sigma\left\{\mathbf{i} \times \left(\frac{\partial \phi}{\partial x}\mathbf{A} + \phi\frac{\partial \mathbf{A}}{\partial x}\right)\right\}$$

$$= \Sigma\left\{\mathbf{i} \times \left(\frac{\partial \phi}{\partial x}\mathbf{A}\right)\right\} + \Sigma\left\{\mathbf{i} \times \left(\phi\frac{\partial \mathbf{A}}{\partial x}\right)\right\}$$

$$= \Sigma\left\{\left(\frac{\partial \phi}{\partial x}\mathbf{i}\right) \times \mathbf{A}\right\} + \Sigma\left\{\phi\left(\mathbf{i} \times \frac{\partial \mathbf{A}}{\partial x}\right)\right\}$$

$$[\because\ \mathbf{a} \times (m\mathbf{b}) = (m\mathbf{a}) \times \mathbf{b} = m(\mathbf{a} \times \mathbf{b})]$$

$$= \left\{\Sigma\left(\frac{\partial \phi}{\partial x}\mathbf{i}\right)\right\} \times \mathbf{A} + \phi\,\Sigma\left(\mathbf{i} \times \frac{\partial \mathbf{A}}{\partial x}\right) = (\nabla\phi) \times \mathbf{A} + \phi(\nabla \times \mathbf{A}).$$

5. *Prove that div* **(A × B) = B •** *curl* **A − A •** *curl* **B**

or $\nabla \bullet (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \bullet (\nabla \times \mathbf{A}) - \mathbf{A} \bullet (\nabla \times \mathbf{B})$.

(Garhwal 2003; Agra 05; Meerut 04, 05B, 08, 09; Bundelkhand 05, 07; Kashi 13; Avadh 09)

Proof: We have

$$\text{div}\,(\mathbf{A} \times \mathbf{B}) = \Sigma\left\{\mathbf{i} \bullet \frac{\partial}{\partial x}(\mathbf{A} \times \mathbf{B})\right\} = \Sigma\left\{\mathbf{i} \bullet \left(\frac{\partial \mathbf{A}}{\partial x} \times \mathbf{B} + \mathbf{A} \times \frac{\partial \mathbf{B}}{\partial x}\right)\right\}$$

$$= \Sigma\left\{\mathbf{i} \bullet \left(\frac{\partial \mathbf{A}}{\partial x} \times \mathbf{B}\right)\right\} + \Sigma\left\{\mathbf{i} \bullet \left(\mathbf{A} \times \frac{\partial \mathbf{B}}{\partial x}\right)\right\}$$

$$= \Sigma\left\{\left(\mathbf{i} \times \frac{\partial \mathbf{A}}{\partial x}\right) \bullet \mathbf{B}\right\} - \Sigma\left\{\mathbf{i} \bullet \left(\frac{\partial \mathbf{B}}{\partial x} \times \mathbf{A}\right)\right\}$$

$$[\because\ \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} \text{ and } \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = -\,\mathbf{a} \bullet (\mathbf{c} \times \mathbf{b})]$$

$$= \left\{ \Sigma \left(\mathbf{i} \times \frac{\partial \mathbf{A}}{\partial x} \right) \right\} \bullet \mathbf{B} - \Sigma \left\{ \left(\mathbf{i} \times \frac{\partial \mathbf{B}}{\partial x} \right) \bullet \mathbf{A} \right\}$$

$$= (\text{curl } \mathbf{A}) \bullet \mathbf{B} - \left\{ \Sigma \left(\mathbf{i} \times \frac{\partial \mathbf{B}}{\partial x} \right) \right\} \bullet \mathbf{A}$$

$$= (\text{curl } \mathbf{A}) \bullet \mathbf{B} - (\text{curl } \mathbf{B}) \bullet \mathbf{A} = \mathbf{B} \bullet \text{curl } \mathbf{A} - \mathbf{A} \bullet \text{curl } \mathbf{B}.$$

6. *Prove that* $\mathit{curl}\ (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \bullet \nabla)\ \mathbf{A} - \mathbf{B}\ \mathit{div}\ \mathbf{A} - (\mathbf{A} \bullet \nabla)\ \mathbf{B} + \mathbf{A}\ \mathit{div}\ \mathbf{B}$.

(Garhwal 2001; Meerut 06B; Kumaun 09, 11, 12)

Proof: We have curl $(\mathbf{A} \times \mathbf{B}) = \nabla \times (\mathbf{A} \times \mathbf{B})$

$$= \Sigma \left\{ \mathbf{i} \times \frac{\partial}{\partial x} (\mathbf{A} \times \mathbf{B}) \right\} = \Sigma \left\{ \mathbf{i} \times \left(\mathbf{A} \times \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \times \mathbf{B} \right) \right\}$$

$$= \Sigma \left\{ \mathbf{i} \times \left(\mathbf{A} \times \frac{\partial \mathbf{B}}{\partial x} \right) \right\} + \Sigma \left\{ \mathbf{i} \times \left(\frac{\partial \mathbf{A}}{\partial x} \times \mathbf{B} \right) \right\}$$

$$= \Sigma \left\{ \left(\mathbf{i} \bullet \frac{\partial \mathbf{B}}{\partial x} \right) \mathbf{A} - (\mathbf{i} \bullet \mathbf{A}) \frac{\partial \mathbf{B}}{\partial x} \right\} + \Sigma \left\{ (\mathbf{i} \bullet \mathbf{B}) \frac{\partial \mathbf{A}}{\partial x} - \left(\mathbf{i} \bullet \frac{\partial \mathbf{A}}{\partial x} \right) \mathbf{B} \right\}$$

$$= \Sigma \left\{ \left(\mathbf{i} \bullet \frac{\partial \mathbf{B}}{\partial x} \right) \mathbf{A} \right\} - \Sigma \left\{ (\mathbf{A} \bullet \mathbf{i}) \frac{\partial \mathbf{B}}{\partial x} \right\}$$

$$+ \Sigma \left\{ (\mathbf{B} \bullet \mathbf{i}) \frac{\partial \mathbf{A}}{\partial x} \right\} - \Sigma \left\{ \left(\mathbf{i} \bullet \frac{\partial \mathbf{A}}{\partial x} \right) \mathbf{B} \right\}$$

$$= \left\{ \Sigma \left(\mathbf{i} \bullet \frac{\partial \mathbf{B}}{\partial x} \right) \right\} \mathbf{A} - \left\{ \mathbf{A} \bullet \Sigma\, \mathbf{i} \frac{\partial}{\partial x} \right\} \mathbf{B} + \left\{ \mathbf{B} \bullet \Sigma\, \mathbf{i} \frac{\partial}{\partial x} \right\} \mathbf{A}$$

$$- \left\{ \Sigma \left(\mathbf{i} \bullet \frac{\partial \mathbf{A}}{\partial x} \right) \right\} \mathbf{B}$$

$$= (\text{div } \mathbf{B})\ \mathbf{A} - (\mathbf{A} \bullet \nabla)\ \mathbf{B} + (\mathbf{B} \bullet \nabla)\ \mathbf{A} - (\text{div } \mathbf{A})\ \mathbf{B}\,.$$

7. *Prove that* $\mathit{grad}\ (\mathbf{A} \bullet \mathbf{B}) = (\mathbf{B} \bullet \nabla)\ \mathbf{A} + (\mathbf{A} \bullet \nabla)\ \mathbf{B} + \mathbf{B} \times \mathit{curl}\ \mathbf{A} + \mathbf{A} \times \mathit{curl}\ \mathbf{B}$.

(Agra 2001; Kumaun 15)

Proof: We have

$$\text{grad } (\mathbf{A} \bullet \mathbf{B}) = \nabla (\mathbf{A} \bullet \mathbf{B}) = \Sigma\, \mathbf{i} \frac{\partial}{\partial x} (\mathbf{A} \bullet \mathbf{B}) = \Sigma\, \mathbf{i} \left(\mathbf{A} \bullet \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \bullet \mathbf{B} \right)$$

$$= \Sigma \left\{ \left(\mathbf{A} \bullet \frac{\partial \mathbf{B}}{\partial x} \right) \mathbf{i} \right\} + \Sigma \left\{ \left(\mathbf{B} \bullet \frac{\partial \mathbf{A}}{\partial x} \right) \mathbf{i} \right\}. \qquad \ldots(1)$$

Now we know that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\ \mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\ \mathbf{c}$.

$\therefore \qquad (\mathbf{a} \bullet \mathbf{b})\ \mathbf{c} = (\mathbf{a} \bullet \mathbf{c})\ \mathbf{b} - \mathbf{a} \times (\mathbf{b} \times \mathbf{c}).$

$$\therefore \qquad \left(\mathbf{A} \bullet \frac{\partial \mathbf{B}}{\partial x} \right) \mathbf{i} = (\mathbf{A} \bullet \mathbf{i}) \frac{\partial \mathbf{B}}{\partial x} - \mathbf{A} \times \left(\frac{\partial \mathbf{B}}{\partial x} \times \mathbf{i} \right)$$

$$= (\mathbf{A} \bullet \mathbf{i}) \frac{\partial \mathbf{B}}{\partial x} + \mathbf{A} \times \left(\mathbf{i} \times \frac{\partial \mathbf{B}}{\partial x} \right).$$

Thus
$$\Sigma\left\{\left(\mathbf{A}\bullet\frac{\partial \mathbf{B}}{\partial x}\right)\mathbf{i}\right\} = \Sigma\left\{(\mathbf{A}\bullet\mathbf{i})\frac{\partial \mathbf{B}}{\partial x}\right\} + \Sigma\left\{\mathbf{A}\times\left(\mathbf{i}\times\frac{\partial \mathbf{B}}{\partial x}\right)\right\}$$
$$= \left\{\mathbf{A}\bullet\Sigma\,\mathbf{i}\frac{\partial}{\partial x}\right\}\mathbf{B} + \mathbf{A}\times\Sigma\left(\mathbf{i}\times\frac{\partial \mathbf{B}}{\partial x}\right)$$
$$= (\mathbf{A}\bullet\nabla)\,\mathbf{B} + \mathbf{A}\times(\nabla\times\mathbf{B}). \qquad \ldots(2)$$

Similarly
$$\Sigma\left\{\left(\mathbf{B}\bullet\frac{\partial \mathbf{A}}{\partial x}\right)\mathbf{i}\right\} = (\mathbf{B}\bullet\nabla)\,\mathbf{A} + \mathbf{B}\times(\nabla\times\mathbf{A}). \qquad \ldots(3)$$

Putting the values from (2) and (3) in (1), we get
$$\text{grad}\,(\mathbf{A}\bullet\mathbf{B}) = (\mathbf{A}\bullet\nabla)\,\mathbf{B} + \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{B}\bullet\nabla)\,\mathbf{A} + \mathbf{B}\times(\nabla\times\mathbf{A}).$$

Note: If we put **A** in place of **B**, then
$$\text{grad}\,(\mathbf{A}\bullet\mathbf{A}) = 2\,(\mathbf{A}\bullet\nabla)\,\mathbf{A} + 2\mathbf{A}\times(\nabla\times\mathbf{A})$$
or
$$\frac{1}{2}\,\text{grad}\;\mathbf{A}^2 = (\mathbf{A}\bullet\nabla)\,\mathbf{A} + \mathbf{A}\times\text{curl}\;\mathbf{A}.$$

8. *Prove that div grad* $\phi = \nabla^2\phi$ *i.e.,* $\nabla\bullet(\nabla\phi) = \nabla^2\phi$.

Proof: We have
$$\nabla\bullet(\nabla\phi) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\bullet\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$
$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi.$$

9. *Prove that curl of the gradient of* ϕ *is zero i.e.,* $\nabla\times(\nabla\phi) = \mathbf{0}$, *i.e., curl grad* $\phi = \mathbf{0}$.

(Bundelkhand 2014)

Proof: We have grad $\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$.

∴
$$\text{curl grad}\,\phi = \nabla\times\text{grad}\,\phi$$
$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\times\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \end{vmatrix}$$
$$= \left(\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y}\right)\mathbf{i} + \left(\frac{\partial^2\phi}{\partial z\,\partial x} - \frac{\partial^2\phi}{\partial x\,\partial z}\right)\mathbf{j} + \left(\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial y\,\partial x}\right)\mathbf{k}$$

$$= 0\ \mathbf{i} + 0\ \mathbf{j} + 0\ \mathbf{k} = \mathbf{0},$$

provided we suppose that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial.

10. *Prove that div curl* $\mathbf{A} = 0$ *, i.e.,* $\nabla \bullet (\nabla \times \mathbf{A}) = 0$.

(Garhwal 2000; Meerut 02, 06B, 09; Agra 2000; Rohilkhand 05; Bundelkhand 08; Kumaun 14)

Proof: Let $\mathbf{A} = A_1\ \mathbf{i} + A_2\ \mathbf{j} + A_3\ \mathbf{k}$. Then

$$\text{curl}\ \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial y} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \mathbf{k}.$$

Now div curl $\mathbf{A} = \nabla \bullet (\nabla \times \mathbf{A})$

$$= \frac{\partial}{\partial x} \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)$$

$$= \frac{\partial^2 A_3}{\partial x\, \partial y} - \frac{\partial^2 A_2}{\partial x\, \partial z} + \frac{\partial^2 A_1}{\partial y\, \partial z} - \frac{\partial^2 A_3}{\partial y\, \partial x} + \frac{\partial^2 A_2}{\partial z\, \partial x} - \frac{\partial^2 A_1}{\partial z\, \partial y}$$

$= 0$, assuming that $\mathbf{A}$ has continuous second partial derivatives.

11. *Prove that* $\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \bullet \mathbf{A}) - \nabla^2 \mathbf{A}$.

(Meerut 2000, 07B, 10B; Rohilkhand 14; Kumaun 10)

Proof: Let $\mathbf{A} = A_1\ \mathbf{i} + A_2\ \mathbf{j} + A_3\ \mathbf{k}$.

Then $$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \mathbf{k}.$$

$\therefore$ $$\nabla \times (\nabla \times \mathbf{A}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial A_3}{\partial y} - \dfrac{\partial A_2}{\partial z} & \dfrac{\partial A_1}{\partial z} - \dfrac{\partial A_3}{\partial x} & \dfrac{\partial A_2}{\partial x} - \dfrac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \Sigma \left[\left\{ \frac{\partial}{\partial y} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \right\} \mathbf{i} \right]$$

$$= \Sigma \left[\left\{ \left(\frac{\partial^2 A_2}{\partial y\, \partial x} + \frac{\partial^2 A_3}{\partial z\, \partial x} \right) - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} \right) \right\} \mathbf{i} \right]$$

$$= \Sigma \left[\left\{ \frac{\partial}{\partial x} \left(\frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} \right) \right\} \mathbf{i} \right]$$

$$= \Sigma \left[\left\{ \frac{\partial}{\partial x} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) - \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} \right) \right\} \mathbf{i} \right]$$

$$= \Sigma \left[\left\{ \frac{\partial}{\partial x} (\nabla \bullet \mathbf{A}) - (\nabla^2 A_1) \right\} \mathbf{i} \right]$$

$$= \Sigma \left[\left\{ \frac{\partial}{\partial x} (\nabla \bullet \mathbf{A}) \right\} \mathbf{i} \right] - \nabla^2\ \Sigma\ A_1 \mathbf{i} = \nabla\, (\nabla \bullet \mathbf{A}) - \nabla^2 \mathbf{A}.$$

Illustrative Examples

Example 24: *Taking* $\mathbf{F} = x^2 y\mathbf{i} + xz\mathbf{j} + 2yz\mathbf{k}$, *verify that div curl* $\mathbf{F} = 0$.

(Garhwal 2002; Bundelkhand 04; Kanpur 10)

Solution: We have curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & xz & 2yz \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y} (2yz) - \frac{\partial}{\partial z} (xz) \right] \mathbf{i} - \left[\frac{\partial}{\partial x} (2yz) - \frac{\partial}{\partial z} (x^2 y) \right] \mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x} (xz) - \frac{\partial}{\partial y} (x^2 y) \right] \mathbf{k}$$

$$= (2z - x)\, \mathbf{i} - 0\ \mathbf{j} + (z - x^2)\, \mathbf{k} = (2z - x)\, \mathbf{i} + (z - x^2)\, \mathbf{k}\ .$$

Now div curl $\mathbf{F} = \text{div}\ [(2z - x)\, \mathbf{i} + (z - x^2)\, \mathbf{k}]$

$$= \frac{\partial}{\partial x} (2z - x) + \frac{\partial}{\partial z} (z - x^2) = -1 + 1 = 0.$$

Example 25: *Prove that div* $(r^n\ \mathbf{r}) = (n + 3)\, r^n$. **(Bundelkhand 2006)**

Solution: We have

$$\text{div}\ (\phi \mathbf{A}) = \phi\ (\text{div}\ \mathbf{A}) + \mathbf{A} \bullet \text{grad}\ \phi\ .$$

Putting $\mathbf{A} = \mathbf{r}$ and $\phi = r^n$ in this identity, we get

$$\text{div}\,(r^n\,\mathbf{r}) = r^n\,\text{div}\,\mathbf{r} + \mathbf{r} \bullet \text{grad}\,r^n$$

$$= 3r^n + \mathbf{r} \bullet (n\,r^{n-1}\,\text{grad}\,r)$$

$$[\because\ \text{div}\,\mathbf{r} = 3 \text{ and grad } f(u) = f'(u)\text{ grad } u]$$

$$= 3r^n + \mathbf{r} \bullet \left[n\,r^{n-1}\,\frac{1}{r}\,\mathbf{r}\right] \qquad \left[\because\ \text{grad}\,r = \hat{\mathbf{r}} = \frac{1}{r}\,\mathbf{r}\right]$$

$$= 3r^n + n\,r^{n-2}\,(\mathbf{r} \bullet \mathbf{r})$$

$$= 3\,r^n + n\,r^{n-2}\,r^2$$

$$= (n+3)\,r^n.$$

Example 26: *Prove that div* $\left(\dfrac{\mathbf{r}}{r^3}\right) = 0.$ **(Bundelkhand 2010)**

Solution: We have

$$\text{div}\left(\frac{1}{r^3}\,\mathbf{r}\right) = \text{div}\,(r^{-3}\,\mathbf{r}) = r^{-3}\,\text{div}\,\mathbf{r} + \mathbf{r} \bullet \text{grad}\,r^{-3}$$

$$= 3r^{-3} + \mathbf{r} \bullet (-3r^{-4}\,\text{grad}\,r) = 3r^{-3} + \mathbf{r} \bullet \left(-3r^{-4}\,\frac{1}{r}\,\mathbf{r}\right)$$

$$= 3r^{-3} - 3r^{-5}\,(\mathbf{r} \bullet \mathbf{r}) = 3r^{-3} - 3r^{-5}\,r^2$$

$$= 3r^{-3} - 3r^{-3} = 0.$$

$\therefore$ the vector $r^{-3}\,\mathbf{r}$ is solenoidal.

Example 27: *Prove that div* $\hat{\mathbf{r}} = 2/r.$ **(Bundelkhand 2007; Kanpur 05)**

Solution: div $(\hat{\mathbf{r}}) = \text{div}\left(\dfrac{1}{r}\,\mathbf{r}\right)\cdot$ Now proceed as in Example 25.

Alternative Method:

$$\text{div}\,\hat{\mathbf{r}} = \text{div}\left(\frac{1}{r}\,\mathbf{r}\right) = \text{div}\left[\frac{1}{r}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\right]$$

$$= \text{div}\left(\frac{x}{r}\,\mathbf{i} + \frac{y}{r}\,\mathbf{j} + \frac{z}{r}\,\mathbf{k}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \left(\frac{1}{r} - \frac{x}{r^2}\,\frac{\partial r}{\partial x}\right) + \left(\frac{1}{r} - \frac{y}{r^2}\,\frac{\partial r}{\partial y}\right) + \left(\frac{1}{r} - \frac{z}{r^2}\,\frac{\partial r}{\partial z}\right).$$

Now $\qquad r^2 = x^2 + y^2 + z^2.$

$\therefore \qquad 2r\,\dfrac{\partial r}{\partial x} = 2x$ *i.e.*, $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}\cdot$

Similarly $\qquad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}\cdot$

$\therefore \qquad \text{div}\,\hat{\mathbf{r}} = \dfrac{3}{r} - \left(\dfrac{x}{r^2}\,\dfrac{x}{r} + \dfrac{y}{r^2}\,\dfrac{y}{r} + \dfrac{z}{r^2}\,\dfrac{z}{r}\right)$

$$= \frac{3}{r} - \frac{x^2 + y^2 + z^2}{r^3} = \frac{3}{r} - \frac{r^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}.$$

Example 28: *Prove that* $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$. **(Meerut 2003, 05, 06B)**

Solution: We know that if ϕ is a scalar function, then $\nabla^2 \phi = \nabla \bullet (\nabla \phi)$.

$$\therefore \quad \nabla^2 f(r) = \nabla \bullet \{\nabla f(r)\} = \text{div}\,\{\text{grad}\, f(r)\}$$

$$= \text{div}\,\{f'(r)\,\text{grad}\, r\} = \text{div}\left\{\frac{1}{r} f'(r)\,\mathbf{r}\right\}$$

$$= \frac{1}{r} f'(r)\,\text{div}\,\mathbf{r} + \mathbf{r} \bullet \text{grad}\left\{\frac{1}{r} f'(r)\right\}$$

$$= \frac{3}{r} f'(r) + \mathbf{r} \bullet \left[\frac{d}{dr}\left\{\frac{1}{r} f'(r)\right\}\text{grad}\, r\right]$$

$$= \frac{3}{r} f'(r) + \mathbf{r} \bullet \left[\left\{-\frac{1}{r^2} f'(r) + \frac{1}{r} f''(r)\right\}\frac{1}{r}\,\mathbf{r}\right]$$

$$= \frac{3}{r} f'(r) + \left[\frac{1}{r}\left\{-\frac{1}{r^2} f'(r) + \frac{1}{r} f''(r)\right\}\right](\mathbf{r} \bullet \mathbf{r})$$

$$= \frac{3}{r} f'(r) + \left[\frac{1}{r}\left\{-\frac{1}{r^2} f'(r) + \frac{1}{r} f''(r)\right\}\right] r^2$$

$$= \frac{3}{r} f'(r) - \frac{1}{r} f'(r) + f''(r) = f''(r) + \frac{2}{r} f'(r).$$

Example 29: *Prove that* $\nabla^2\left(\frac{1}{r}\right) = 0$ or $\text{div}\left(\textit{grad}\,\frac{1}{r}\right) = 0$.

(Agra 2002; Meerut 07, 13)

Solution: We have

$$\nabla^2\left(\frac{1}{r}\right) = \nabla \bullet \left(\nabla \frac{1}{r}\right) = \text{div}\left(\text{grad}\,\frac{1}{r}\right)$$

$$= \text{div}\left(-\frac{1}{r^2}\,\text{grad}\, r\right) = \text{div}\left(-\frac{1}{r^2}\frac{1}{r}\,\mathbf{r}\right) = \text{div}\left(-\frac{1}{r^3}\,\mathbf{r}\right)$$

$$= \left(-\frac{1}{r^3}\right)\text{div}\,\mathbf{r} + \mathbf{r} \bullet \text{grad}\left(-\frac{1}{r^3}\right) = -\frac{3}{r^3} + \mathbf{r} \bullet \left[\frac{d}{dr}\left(-\frac{1}{r^3}\right)\text{grad}\, r\right]$$

$$= -\frac{3}{r^3} + \mathbf{r} \bullet \left(\frac{3}{r^4}\frac{1}{r}\,\mathbf{r}\right) = -\frac{3}{r^3} + \frac{3}{r^5}(\mathbf{r} \bullet \mathbf{r}) = -\frac{3}{r^3} + \frac{3}{r^5} r^2 = 0.$$

$\therefore$ $1/r$ is a solution of Laplace's equation.

Example 30: *Prove that div grad* $r^n = n(n+1) r^{n-2}$, *i.e.,* $\nabla^2 r^n = n(n+1) r^{n-2}$.

(Bundelkhand 2005, 10; Meerut 2000, 08; Avadh 10; Agra 14)

Solution: We have $\nabla^2 r^n = \nabla \bullet (\nabla r^n) = \text{div}\,(\text{grad}\, r^n)$

$$= \text{div } (nr^{n-1} \text{ grad } r) = \text{div } \left(nr^{n-1} \frac{1}{r} \mathbf{r} \right) = \text{div } (nr^{n-2} \ \mathbf{r})$$

$$= (nr^{n-2}) \text{ div } \mathbf{r} + \mathbf{r} \bullet (\text{grad } nr^{n-2})$$

$$= 3nr^{n-2} + \mathbf{r} \bullet [n\,(n-2)\, r^{n-3} \text{ grad } r]$$

$$= 3nr^{n-2} + \mathbf{r} \bullet \left[n\,(n-2)\, r^{n-3} \frac{1}{r} \mathbf{r} \right]$$

$$= 3nr^{n-2} + \mathbf{r} \bullet [n\,(n-2)\, r^{n-4} \ \mathbf{r}] = 3nr^{n-2} + n\,(n-2)\, r^{n-4} \ (\mathbf{r} \bullet \mathbf{r})$$

$$= 3nr^{n-2} + n\,(n-2)\, r^{n-4} \ r^2 = nr^{n-2} \ (3 + n - 2) = n\,(n+1)\, r^{n-2}.$$

Note: If $n = -1$, then $\nabla^2 (r^{-1}) = (-1)(-1+1)\, r^{-3} = 0$.

Example 31: *Prove that* $\nabla \bullet \left\{ r \, \nabla \left(\frac{1}{r^3} \right) \right\} = \frac{3}{r^4}$ *or,* $\text{div } [r \text{ grad } r^{-3}] = 3r^{-4}$. **(Meerut 2009B)**

Solution: We have $\nabla \left(\frac{1}{r^3} \right) = \text{ grad } r^{-3}$

$$= \frac{\partial}{\partial x} (r^{-3})\, \mathbf{i} + \frac{\partial}{\partial y} (r^{-3})\, \mathbf{j} + \frac{\partial}{\partial z} (r^{-3})\, \mathbf{k}\,.$$

Now $\frac{\partial}{\partial x} (r^{-3}) = -\,3r^{-4} \frac{\partial r}{\partial x} \cdot$

But $r^2 = x^2 + y^2 + z^2$.

Therefore $2r \frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r} \cdot$

So $\frac{\partial}{\partial x} (r^{-3}) = -\,3r^{-4} \frac{x}{r} = -\,3r^{-5} x$.

Similarly $\frac{\partial}{\partial y} (r^{-3}) = -\,3r^{-5} y$ and $\frac{\partial}{\partial z} (r^{-3}) = -\,3r^{-5} z$.

Therefore $\nabla \left(\frac{1}{r^3} \right) = -\,3r^{-5} \ (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.

$\therefore$ $r \, \nabla \left(\frac{1}{r^3} \right) = -\,3r^{-4} \ (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.

$\therefore$ $\nabla \bullet \left(r \, \nabla \frac{1}{r^3} \right) = \frac{\partial}{\partial x} (-3r^{-4} x) + \frac{\partial}{\partial y} (-\,3r^{-4} y) + \frac{\partial}{\partial z} (-3r^{-4} z)$.

Now $\frac{\partial}{\partial x} (-3r^{-4} \ x) = 12r^{-5} \frac{\partial r}{\partial x} x - 3r^{-4}$

$$= 12r^{-5} \frac{x}{r} x - 3r^{-4} = 12r^{-6} x^2 - 3r^{-4}.$$

Similarly $\frac{\partial}{\partial y} (-3r^{-4} y) = 12r^{-6} y^2 - 3r^{-4}$

and $\frac{\partial}{\partial z} (-3r^{-4} \ z) = 12r^{-6} \ z^2 - 3r^{-4}$.

Hence $$\nabla \bullet \left(r \nabla \frac{1}{r^3} \right) = 12\, r^{-6}\ (x^2 + y^2 + z^2) - 9r^{-4}$$

$$= 12\, r^{-6} r^2 - 9r^{-4} = 12\, r^{-4} - 9r^{-4} = 3r^{-4}.$$

Example 32: *If* **a** *is a constant vector, prove that div* $\{r^n\ (\mathbf{a} \times \mathbf{r})\} = 0$ **(Kanpur 2008)**

Solution: We have div $(\phi \mathbf{A}) = \phi$ div $\mathbf{A} + \mathbf{A} \bullet$ grad ϕ.

$\therefore$ $$\text{div}\ \{r^n\ (\mathbf{a} \times \mathbf{r})\} = r^n\ \text{div}\ (\mathbf{a} \times \mathbf{r}) + (\mathbf{a} \times \mathbf{r}) \bullet \text{grad}\ r^n$$

$$= r^n\ \text{div}\ (\mathbf{a} \times \mathbf{r}) + (\mathbf{a} \times \mathbf{r}) \bullet (nr^{n-1}\ \text{grad}\ r)$$

$$= r^n\ (\mathbf{r} \bullet \text{curl}\ \mathbf{a} - \mathbf{a} \bullet \text{curl}\ \mathbf{r}) + (\mathbf{a} \times \mathbf{r}) \bullet \left(nr^{n-1}\ \frac{1}{r}\ \mathbf{r} \right)$$

$$= r^n\ (\mathbf{r} \bullet \mathbf{0} - \mathbf{a} \bullet \mathbf{0}) + nr^{n-2}\ (\mathbf{a} \times \mathbf{r}) \bullet \mathbf{r}$$

[$\because$ curl of a constant vector is zero and curl $\mathbf{r} = \mathbf{0}$]

$$= nr^{n-2}\ [\mathbf{a}, \mathbf{r}, \mathbf{r}]$$

$= 0$, since a scalar triple product having two equal vectors is zero.

Example 33: *If* **a** *and* **b** *are constant vectors, prove that*

(i) *div* $[(\mathbf{r} \times \mathbf{a}) \times \mathbf{b}] = -2\mathbf{b} \bullet \mathbf{a}$. **(Bundelkhand 2010)**

(ii) *curl* $[(\mathbf{r} \times \mathbf{a}) \times \mathbf{b}] = \mathbf{b} \times \mathbf{a}$. **(Bundelkhand 2010)**

Solution: (i) We have $(\mathbf{r} \times \mathbf{a}) \times \mathbf{b} = (\mathbf{b} \bullet \mathbf{r})\ \mathbf{a} - (\mathbf{b} \bullet \mathbf{a})\ \mathbf{r}$.

$\therefore$ $$\text{div}\ [(\mathbf{r} \times \mathbf{a}) \times \mathbf{b}] = \text{div}\ [(\mathbf{b} \bullet \mathbf{r})\ \mathbf{a} - (\mathbf{b} \bullet \mathbf{a})\ \mathbf{r}]$$

$$= \text{div}\ [(\mathbf{b} \bullet \mathbf{r})\ \mathbf{a}] - \text{div}\ [(\mathbf{b} \bullet \mathbf{a})\ \mathbf{r}] \qquad \ldots(1)$$

But div $(\phi \mathbf{A}) = \phi$ div $\mathbf{A} + \mathbf{A} \bullet$ grad ϕ.

Taking $\phi = \mathbf{b} \bullet \mathbf{r}$ and $\mathbf{A} = \mathbf{a}$, we get

$$\text{div}\ [(\mathbf{b} \bullet \mathbf{r})\ \mathbf{a}] = (\mathbf{b} \bullet \mathbf{r})\ \text{div}\ \mathbf{a} + \mathbf{a} \bullet \text{grad}\ (\mathbf{b} \bullet \mathbf{r}).$$

Since **a** is a constant vector, therefore div $\mathbf{a} = 0$.

Also let $\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}$.

Then $\mathbf{b} \bullet \mathbf{r} = (b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}) \bullet (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$= b_1 x + b_2 y + b_3 z$, where b_1, b_2, b_3 are constants.

$\therefore$ grad $(\mathbf{b} \bullet \mathbf{r}) = b_1\ \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} = \mathbf{b}$.

$\therefore$ $$\text{div}\ [(\mathbf{b} \bullet \mathbf{r})\ \mathbf{a}] = \mathbf{a} \bullet \mathbf{b}. \qquad \ldots(2)$$

Again div $[(\mathbf{b} \bullet \mathbf{a})]\ \mathbf{r} = (\mathbf{b} \bullet \mathbf{a})$ div $\mathbf{r} + \mathbf{r} \bullet$ grad $(\mathbf{b} \bullet \mathbf{a})$.

But div $\mathbf{r} = 3$.

Also grad $(\mathbf{b} \bullet \mathbf{a}) = \mathbf{0}$ because $\mathbf{b} \bullet \mathbf{a}$ is constant.

$\therefore$ $$\text{div}\ [(\mathbf{b} \bullet \mathbf{a})\ \mathbf{r}] = 3\ (\mathbf{b} \bullet \mathbf{a}). \qquad \ldots(3)$$

Substituting the values from (2) and (3) in (1), we get

$$\text{div}\ [(\mathbf{r} \times \mathbf{a}) \times \mathbf{b}] = (\mathbf{a} \bullet \mathbf{b}) - 3\ (\mathbf{b} \bullet \mathbf{a}) = -2\mathbf{b} \bullet \mathbf{a}.$$

(ii) $$\text{curl}\ [(\mathbf{r} \times \mathbf{a}) \times \mathbf{b}] = \text{curl}\ [(\mathbf{b} \bullet \mathbf{r})\ \mathbf{a} - (\mathbf{b} \bullet \mathbf{a})\ \mathbf{r}]$$

$$= \text{curl}\ [(\mathbf{b} \bullet \mathbf{r})\ \mathbf{a}] - \text{curl}\ [(\mathbf{b} \bullet \mathbf{a})\ \mathbf{r}].$$

But $\operatorname{curl}(\phi\mathbf{A}) = \operatorname{grad}\phi \times \mathbf{A} + \phi \operatorname{curl}\mathbf{A}$.

$\therefore$ $\operatorname{curl}[(\mathbf{b}\bullet\mathbf{r})\,\mathbf{a}] = [\operatorname{grad}(\mathbf{b}\bullet\mathbf{r})]\times\mathbf{a} + (\mathbf{b}\bullet\mathbf{r})\operatorname{curl}\mathbf{a} = \mathbf{b}\times\mathbf{a}$

[$\because$ curl $\mathbf{a} = \mathbf{0}$ and grad $(\mathbf{b}\bullet\mathbf{r}) = \mathbf{b}$]

Also $\operatorname{curl}[(\mathbf{b}\bullet\mathbf{a})\,\mathbf{r}] = [\operatorname{grad}(\mathbf{b}\bullet\mathbf{a})]\times\mathbf{r} + (\mathbf{b}\bullet\mathbf{a})\operatorname{curl}\mathbf{r} = \mathbf{0}$

[$\because$ grad $(\mathbf{b}\bullet\mathbf{a}) = \mathbf{0}$ and curl $\mathbf{r} = \mathbf{0}$]

$\therefore$ $\operatorname{curl}[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}] = \mathbf{b}\times\mathbf{a} - \mathbf{0} = \mathbf{b}\times\mathbf{a}$.

Example 34: *Prove that curl* $[\mathbf{r}\times(\mathbf{a}\times\mathbf{r})] = 3\mathbf{r}\times\mathbf{a}$, *where* $\mathbf{a}$ *is a constant vector.*

(Purvanchal 2009)

Solution: $\operatorname{curl}[\mathbf{r}\times(\mathbf{a}\times\mathbf{r})]$

$= \nabla\times[(\mathbf{r}\bullet\mathbf{r})\,\mathbf{a} - (\mathbf{r}\bullet\mathbf{a})\,\mathbf{r}]$ [$\because$ $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\bullet\mathbf{c})\,\mathbf{b} - (\mathbf{a}\bullet\mathbf{b})\,\mathbf{c}$]

$= \nabla\times[r^2\mathbf{a} - (\mathbf{r}\bullet\mathbf{a})\,\mathbf{r}]$ [$\because$ $\mathbf{r}\bullet\mathbf{r} = \mathbf{r}^2 = r^2$]

$= \nabla\times(r^2\mathbf{a}) - \nabla\times[(\mathbf{r}\bullet\mathbf{a})\,\mathbf{r}]$ [$\because$ $\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$]

$= (\nabla r^2)\times\mathbf{a} + r^2\,(\nabla\times\mathbf{a}) - [\nabla(\mathbf{r}\bullet\mathbf{a})]\times\mathbf{r} - (\mathbf{r}\bullet\mathbf{a})\,(\nabla\times\mathbf{r})$

[$\because$ $\nabla\times(\phi\mathbf{A}) = (\nabla\phi)\times\mathbf{A} + \phi\,(\nabla\times\mathbf{A})$]

$= (2r\,\nabla r)\times\mathbf{a} + r^2\,\mathbf{0} - [\nabla(\mathbf{r}\bullet\mathbf{a})]\times\mathbf{r} - (\mathbf{r}\bullet\mathbf{a})\,\mathbf{0}$

[$\because$ $\nabla f(r) = f'(r)\,\nabla r$; $\nabla\times\mathbf{a} = \mathbf{0}$, $\mathbf{a}$ being a constant vector ; and $\nabla\times\mathbf{r} = \mathbf{0}$]

$= \left(2r\,\frac{1}{r}\,\mathbf{r}\right)\times\mathbf{a} - [\nabla(\mathbf{r}\bullet\mathbf{a})]\times\mathbf{r}$

$= 2\mathbf{r}\times\mathbf{a} - \mathbf{a}\times\mathbf{r}$ [$\because$ $\nabla(\mathbf{r}\bullet\mathbf{a}) = \mathbf{a}$, if $\mathbf{a}$ is a constant vector]

$= 2\mathbf{r}\times\mathbf{a} + \mathbf{r}\times\mathbf{a} = 3\mathbf{r}\times\mathbf{a}$.

Example 35: *If* $\mathbf{a}$ *is a constant vector, prove that*

$$\textit{curl}\ \frac{\mathbf{a}\times\mathbf{r}}{r^3} = -\frac{\mathbf{a}}{r^3} + \frac{3\mathbf{r}}{r^5}\,(\mathbf{a}\bullet\mathbf{r}).$$

(Meerut 2009B; Bundelkhand 09, 11)

Solution: We have

$$\operatorname{curl}\frac{\mathbf{a}\times\mathbf{r}}{r^3} = \nabla\times\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = \Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)\right\}.$$

Now
$$\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = -\frac{3}{r^4}\frac{\partial r}{\partial x}(\mathbf{a}\times\mathbf{r}) + \frac{1}{r^3}\left(\mathbf{a}\times\frac{\partial\mathbf{r}}{\partial x}\right) + \frac{1}{r^3}\left(\frac{\partial\mathbf{a}}{\partial x}\times\mathbf{r}\right) \quad \ldots(1)$$

Now $\dfrac{\partial\mathbf{a}}{\partial x} = \mathbf{0}$ because $\mathbf{a}$ is a constant vector.

Also $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

$\therefore$ $\dfrac{\partial\mathbf{r}}{\partial x} = \mathbf{i}$.

Further $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$.

$\therefore$ (1) becomes $$\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\frac{3}{r^4}\ \frac{x}{r}\,(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}\,(\mathbf{a}\times\mathbf{i})$$

$$=-\frac{3x}{r^5}\,(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}\,(\mathbf{a}\times\mathbf{i}).$$

$\therefore$ $$\mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\frac{3x}{r^5}\,\mathbf{i}\times(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}\,\mathbf{i}\times(\mathbf{a}\times\mathbf{i})$$

$$=-\frac{3x}{r^5}\left[(\mathbf{i}\bullet\mathbf{r})\,\mathbf{a}-(\mathbf{i}\bullet\mathbf{a})\,\mathbf{r}\right]+\frac{1}{r^3}\,[(\mathbf{i}\bullet\mathbf{i})\,\mathbf{a}\ -(\mathbf{i}\bullet\mathbf{a})\,\mathbf{i}]$$

$$=-\frac{3x}{r^5}\,x\mathbf{a}+\frac{3x}{r^5}\,a_1\,\mathbf{r}+\frac{1}{r^3}\,\mathbf{a}-\frac{1}{r^3}\,a_1\ \mathbf{i}$$

$[\because\ \mathbf{i}\bullet\mathbf{r}=x$ and $\mathbf{i}\bullet\mathbf{a}=a_1$ if $\mathbf{a}=a_1\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k}]$

$$=-\frac{3x^2}{r^5}\,\mathbf{a}+\frac{3}{r^5}\,a_1 x\mathbf{r}+\frac{1}{r^3}\,\mathbf{a}-\frac{1}{r^3}\,a_1\mathbf{i}\,.$$

$\therefore$ $$\Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)\right\}$$

$$=\left\{-\frac{3}{r^5}\,\Sigma\,x^2\right\}\mathbf{a}+\left\{\frac{3}{r^5}\,\Sigma\,a_1\ x\right\}\mathbf{r}+\frac{3}{r^3}\,\mathbf{a}-\frac{1}{r^3}\,\Sigma\,a_1\mathbf{i}$$

$$=-\frac{3}{r^5}\,r^2\ \mathbf{a}+\frac{3}{r^5}\,(\mathbf{r}\bullet\mathbf{a})\,\mathbf{r}+\frac{3}{r^3}\,\mathbf{a}-\frac{1}{r^3}\,\mathbf{a}$$

$[\because\ \Sigma\,x^2=r^2,\ \Sigma\,a_1 x=\mathbf{r}\bullet\mathbf{a},\ \Sigma\,a_1\mathbf{i}=\mathbf{a}]$

$$=-\frac{\mathbf{a}}{r^3}+\frac{3}{r^5}\,(\mathbf{a}\bullet\mathbf{r})\,\mathbf{r}.$$

Comprehensive Exercise 4

1. (i) Verify that curl grad $f=\mathbf{0}$, where $f=x^2y+2xy+z^2$.

(ii) Prove that grad $f(u)=f'(u)$ grad u.

(iii) Find $\nabla\phi$ and $|\nabla\phi|$ when $\phi=(x^2+y^2+z^2)\,e^{-(x^2+y^2+z^2)^{1/2}}$.

(Kumaun 2012)

2. (i) Prove that curl $(\psi\nabla\phi)=\nabla\psi\times\nabla\phi=-$ curl $(\phi\nabla\psi)$.

(ii) Prove that $\nabla^2(\phi\psi)=\phi\nabla^2\psi+2\nabla\phi\bullet\nabla\psi+\psi\nabla^2\phi$. **(Kumaun 2014)**

(iii) Prove that div $(\nabla\phi\times\nabla\psi)=0$.

(iv) If **A** and **B** are irrotational, prove that **A** × **B** is solenoidal.

(v) Prove that curl (ϕ grad ϕ) $=\mathbf{0}$.

3. (i) Prove that $\mathbf{a}\bullet\{\nabla(\mathbf{v}\bullet\mathbf{a})-\nabla\times(\mathbf{v}\times\mathbf{a})\}=$ div **v**, where **a** is a constant unit vector.

(ii) Prove that vector $f(r)\,\mathbf{r}$ is irrotational.

(iii) If f and g are two scalar point functions, prove that

$$\text{div}\,(f\,\nabla g) = f\,\nabla^2 g + \nabla f \bullet \nabla g.$$

(iv) Show that curl $(\mathbf{a} \bullet \mathbf{r})\,\mathbf{a} = \mathbf{0}$, where $\mathbf{a}$ is a constant vector.

(Kumaun 2007, 09)

[**Hint:** Use identity 4. Note that $\nabla\,(\mathbf{a} \bullet \mathbf{r}) = \mathbf{a}$, if $\mathbf{a}$ is a constant vector.]

4. If $\mathbf{a}$ is a constant vector, then prove that

(i) $\nabla\,(\mathbf{a} \bullet \mathbf{u}) = (\mathbf{a} \bullet \nabla)\,\mathbf{u} + \mathbf{a} \times \text{curl}\,\mathbf{u}$,

(ii) $\nabla \bullet (\mathbf{a} \times \mathbf{u}) = -\,\mathbf{a} \bullet \text{curl}\,\mathbf{u}$,

(iii) $\nabla \times (\mathbf{a} \times \mathbf{u}) = \mathbf{a}\,\text{div}\,\mathbf{u} - (\mathbf{a} \bullet \nabla)\,\mathbf{u}$,

5. (i) Given that $\rho\,\mathbf{F} = \nabla p$, where ρ, p, $\mathbf{F}$ are point functions, prove that

$$\mathbf{F} \bullet \text{curl}\,\mathbf{F} = 0.$$

(ii) A vector function $\mathbf{f}$ is the product of a scalar function and the gradient of a scalar function. Show that $\mathbf{f} \bullet \text{curl}\,\mathbf{f} = 0$.

(iii) Show that curl $\mathbf{a}\,\phi\,(r) = \dfrac{1}{r}\,\phi'(r)\,\mathbf{r} \times \mathbf{a}$, where $\mathbf{a}$ is a constant vector.

6. (i) Prove that curl $[r^n\,(\mathbf{a} \times \mathbf{r})] = (n+2)\,r^n\,\mathbf{a} - nr^{n-2}\,(\mathbf{r} \bullet \mathbf{a})\,\mathbf{r}$,

where $\mathbf{a}$ is a constant vector. **(Kumaun 2008)**

(ii) Prove that $\nabla^2\,(r^n\,\mathbf{r}) = n\,(n+3)\,r^{n-2}\,\mathbf{r}$.

7. (i) Prove that curl grad $r^n = \mathbf{0}$. **(Avadh 2010; Kanpur 11)**

(ii) If $\nabla^2 f\,(r) = 0$, show that $f\,(r) = \dfrac{c_1}{r} + c_2$,

where $r^2 = x^2 + y^2 + z^2$ and c_1, c_2 are arbitrary constants.

8. If $\mathbf{r}$ is the position vector of the point (x, y, z) show that curl $(r^n\,\mathbf{r}) = \mathbf{0}$, where r is the module of $\mathbf{r}$.

9. Prove that $r^n\,\mathbf{r}$ is an irrotational vector for any value of n but is solenoidal only if $n + 3 = 0$.

10. (i) If $\mathbf{u} = (1/r)\,\mathbf{r}$, show that $\nabla \times \mathbf{u} = \mathbf{0}$. **(Avadh 2010)**

(ii) Prove that div $(\mathbf{A} \times \mathbf{r}) = \mathbf{r} \bullet \text{curl}\,\mathbf{A}$.

11. (i) If $\nabla^2 f\,(r) = 0$ show that $f(r) = c_1 \log r + c_2$ where $r^2 = x^2 + y^2$ and c_1, c_2 are arbitrary constants.

(ii) If $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, then show that $\nabla \bullet (\mathbf{a} \bullet \mathbf{b}\mathbf{r}) = 3\mathbf{a} \bullet \mathbf{b}$.

12. If $\mathbf{u} = (1/r)\,\mathbf{r}$ find grad (div $\mathbf{u}$).

13. (i) Prove that $\nabla^2 \left[\nabla \bullet \left(\dfrac{\mathbf{r}}{r^2}\right)\right] = 2r^{-4}$.

(Kumaun 2011, 13)

(ii) Prove that $\mathbf{a} \bullet \left(\nabla \dfrac{1}{r}\right) = -\dfrac{\mathbf{a} \bullet \mathbf{r}}{r^3}$.

(iii) Prove that $\nabla \bullet (U \nabla V - V \nabla U) = U \nabla^2 V - V \nabla^2 U$.

(iv) Prove that $\mathbf{b} \bullet \nabla \left(\mathbf{a} \bullet \nabla \dfrac{1}{r} \right) = \dfrac{3 (\mathbf{a} \bullet \mathbf{r}) (\mathbf{b} \bullet \mathbf{r})}{r^5} - \dfrac{\mathbf{a} \bullet \mathbf{b}}{r^3}$,

where **a** and **b** are constant vectors. **(Kumaun 2015)**

14. Evaluate div $\{\mathbf{a} \times (\mathbf{r} \times \mathbf{a})\}$, where **a** is a constant vector. **(Kanpur 2007)**

15. (i) Prove that div $\left\{ \dfrac{f(r)\,\mathbf{r}}{r} \right\} = \dfrac{1}{r^2} \dfrac{d}{dr} (r^2 f)$. **(Kumaun 2007, 14)**

(ii) Prove that $\dfrac{1}{2} \nabla \mathbf{a}^2 = (\mathbf{a} \bullet \nabla)\, \mathbf{a} + \mathbf{a} \times \text{curl}\, \mathbf{a}$.

16. Prove that curl $[\mathbf{r} \times (\mathbf{a} \times \mathbf{r})] = 3\mathbf{r} \times \mathbf{a}$, where **a** is a constant vector.

17. Prove that $\nabla \times (\mathbf{F} \times \mathbf{r}) = 2\mathbf{F} - (\nabla \bullet \mathbf{F})\, \mathbf{r} + (\mathbf{r} \bullet \nabla)\, \mathbf{F}$. **(Kumaun 2015)**

18. If **a** and **b** are constant vectors, prove that

grad $[(\mathbf{r} \times \mathbf{a}) \bullet (\mathbf{r} \times \mathbf{b})] = (\mathbf{b} \times \mathbf{r}) \times \mathbf{a} + (\mathbf{a} \times \mathbf{r}) \times \mathbf{b}$. **(Kumaun 2015)**

Answers 4

1. (iii) $(2 - r)\, e^{-r}\, \mathbf{r}$; $(2 - r)\, e^{-r}\, r$

12. $-\dfrac{2}{r^3}\, \mathbf{r}$

14. $2\mathbf{a}^2$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If the vector $\mathbf{V} = (x + 3y)\, \mathbf{i} + (y - 2z)\, \mathbf{j} + (x + az)\, \mathbf{k}$ is solenoidal, then the constant a is

(a) 0 (b) 1

(c) –2 (d) 2 **(Garhwal 2006)**

2. The directional derivative of $\phi(x, y, z) = x^2 yz + 4xz^2$ at $(1, -2, -1)$ in the direction of the vector $2\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}$ is

(a) 37/3 (b) 3/37

(c) 3 (d) none of these

3. If $r = |\mathbf{r}|$ where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, then $\nabla^2\, r^n =$

(a) $n(n+1)\, r^n$ (b) $n(n+1)\, r^{n-1}$

(c) $n(n+1)\, r^{n-2}$ (d) none of these

(Agra 2007; Garhwal 15)

4. $\nabla^2 \left(\frac{1}{r}\right) =$

(a) $-2/r^3$ (b) 0

(c) $2/r^3$ (d) none of these

(Kumaun 2014)

5. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{a}$ is a constant vector, then curl $(\mathbf{r} \times \mathbf{a})$ is

(a) $-\mathbf{a}$ (b) $-3\mathbf{a}$

(c) $-2\mathbf{a}$ (d) none of these

(Garhwal2014)

6. The value of div $\hat{\mathbf{r}}$ is

(a) $2/r$ (b) 0

(c) $1/r$ (d) none of these **(Agra 2014)**

7. The $\mathbf{V}$ is a constant vector, then div $\mathbf{V}$ is :

(a) 3 (b) $3\mathbf{V}$

(c) 0 (d) none of these

(Kumaun 2007, 11)

8. If $\mathbf{f} = x^2 y\mathbf{i} - 2xz\mathbf{j} + 2yz\mathbf{k}$, then div $\mathbf{f}$ is equal to

(a) $2x(x+1)$ (b) $2y(x+1)$

(c) $y(y+1)$ (d) none of these

(Kumaun 2008)

9. $\nabla \times (\nabla f)$ is equal to

(a) $\nabla^2 f$ (b) $\mathbf{0}$

(c) 0 (d) none of these

(Kumaun 2010)

10. grad $(1/r)$ is equal to

(a) $-\boldsymbol{r}/r^3$ (b) $\boldsymbol{r}/r^3$

(c) $\boldsymbol{r}/r^2$ (d) none of these

(Kumaun 2006)

11. If $\boldsymbol{f} = \frac{\boldsymbol{r}}{r}$, then curl $\boldsymbol{f}$ is equal to

(a) $|\boldsymbol{f}|$ (b) $|\boldsymbol{f}|^2$

(c) $\mathbf{0}$ (d) none of these

(Kumaun 2006)

12. Value of $(2\mathbf{i}\times\mathbf{j})\times(\mathbf{j}-\mathbf{k})$ is equal to

(a) $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ (b) $-\mathbf{i}+2\mathbf{j}+2\mathbf{k}$

(c) $2\mathbf{i}-2\mathbf{j}+\mathbf{k}$ (d) $2\mathbf{i}+2\mathbf{j}+\mathbf{k}$

(Kumaun 2014)

13. The value of curl $(\mathbf{i}\times\mathbf{j}) =$

(a) $\mathbf{0}$ (b) 1

(c) -1 (d) 0

(Kumaun 2014)

14. If $\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$, then curl $\mathbf{r}$ is

(a) 1 (b) 2

(c) 3 (d) 4

(Garhwal 2007)

15. If $\mathbf{F} = xy^2\,\mathbf{i}+2x^2yz\,\mathbf{j}-3yz^2\,\mathbf{k}$, then div $\mathbf{F}$ at $(1,-1,\ 1)$ is

(a) -9 (b) 9

(c) 8 (d) 10

(Garhwal 2008)

16. For any vector field $\mathbf{F}$, the value of div curl $\mathbf{A} =$

(a) 1 (b) -1

(c) 2 (d) none of these

(Garhwal 2008)

17. The equation of normal plane is

(a) $(\mathbf{R}-\mathbf{r})\times\nabla f = \mathbf{0}$ (b) $(\mathbf{R}+\mathbf{r})\times\nabla f = \mathbf{0}$

(c) $(\mathbf{R}-\mathbf{r})\times\nabla^2 f = \mathbf{0}$ (d) $(\mathbf{R}+\mathbf{r})\times\nabla^2 f = \mathbf{0}$

(Garhwal 2009)

18. If $\phi = 3x^2y - y^3z^2$, then $\nabla\phi$ at the point $(1,-2,-1)$ is

(a) $-12\mathbf{i}-9\mathbf{j}-24\mathbf{k}$ (b) -25

(c) $12\mathbf{i}+9\mathbf{j}-2\mathbf{k}$ (d) none of these

(Garhwal 2011)

19. If for any vector $\mathbf{V}$, then div $\mathbf{V} = 0$, then $\mathbf{V}$ is

(a) solenoidal (b) irrotational

(c) normal to the surface (d) parallel to line surface

(Garhwal 2011)

20. Value of curl grad ϕ is

(a) $\mathbf{0}$ (b) $\nabla^2\phi$

(c) $\frac{1}{2}\nabla^2\phi$ (d) none of these

(Garhwal 2013)

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. If $\mathbf{F} = (x^2 + y^2)\,\mathbf{i} - 2\,xy\,\mathbf{j}$, then $\mathbf{F} \bullet d\mathbf{r} = \ldots\ldots$.
2. If $\mathbf{P} = e^{xy}\,\mathbf{i} + (x - 2\,y)\,\mathbf{j} + (x \sin y)\,\mathbf{k}$, then $\dfrac{\partial \mathbf{P}}{\partial x} = \ldots\ldots$.
3. If **a** is a constant vector then grad $(\mathbf{a} \bullet \mathbf{r}) = \ldots\ldots$.
   (Rohilkhand 2007; Kumaun 12)
4. If **a** is a constant vector, then $\nabla \bullet (\mathbf{a} \times \mathbf{r}) = \ldots\ldots$.
5. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then the value of div $\mathbf{r} = \ldots\ldots$.
   (Agra 2008; Kumaun 15)
6. If $\mathbf{A} = x^2 z\mathbf{i} - 2y^3 z^2\,\mathbf{j} + xy^2 z\mathbf{k}$, then div **A** at $(1, -1, 1) = \ldots\ldots$.
7. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then the value of curl $\mathbf{r} = \ldots\ldots$.
   (Agra 2008; Rohilkhand 14)
8. For any vector **A**, div curl $\mathbf{A} = \ldots\ldots$.
9. A vector **V** is said to be solenoidal if …… . **(Rohilkhand 2005; Agra 06)**
10. A vector **F** is said to be irrotational if …… .
11. If $\phi = x^2 y + 2xy + z^2$, then curl grad $\phi = \ldots\ldots$. **(Bundelkhand 2007)**
12. If $\mathbf{f} = x^2 y\mathbf{i} + 2xz\mathbf{j} + 2yz\mathbf{k}$, then div (curl $\mathbf{f}$) $= \ldots\ldots$. **(Kumaun 2010, 12)**
13. Value of div grad ϕ is ……. **(Kumaun 2011)**
14. If **a** is a constant vector, then curl $(\mathbf{r} \times \mathbf{a}) = \ldots\ldots$. **(Kumaun 2013)**

True or False

Write 'T' for true and 'F' for false statement.

1. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then **r** is solenoidal.
2. If **V** is a constant vector, then div $\mathbf{V} = 0$. **(Rohilkhand 2005)**
3. If $\mathbf{F} = 2xyz\,\mathbf{i} + y^2 z\mathbf{j} - 2\,yz^2\,\mathbf{k}$, then **F** is irrotational.
4. If ϕ is a differentiable scalar function, then curl grad $\phi = \mathbf{0}$.
5. If ϕ is a differentiable scalar function then div grad $\phi = \nabla^2 \phi$.
6. $\nabla \bullet (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \bullet (\nabla \times \mathbf{B}) - \mathbf{B} \bullet (\nabla \times \mathbf{A})$.
7. A function which satisfies Laplace's equation is called a harmonic function.

Answers

Multiple Choice Questions

1.	(c)	2.	(a)	3.	(c)	4.	(b)	5.	(c)
6.	(a)	7.	(c)	8.	(b)	9.	(b)	10.	(a)

11. (c) **12.** (b) **13.** (a) **14.** (d) **15.** (b)
16. (d) **17.** (a) **18.** (d) **19.** (a) **20.** (a)

Fill in the Blank(s)

1. $(x^2 + y^2)\,dx - 2xy\,dy$ **2.** $y\,e^{xy}\,\mathbf{i} + \mathbf{j} + \sin y\,\mathbf{k}$ **3.** $\mathbf{a}$
4. 0 **5.** 3 **6.** –3 **7.** 0 **8.** 0
9. $\text{div}\,\mathbf{V} = 0$ **10.** $\text{curl}\,\mathbf{F} = \mathbf{0}$ **11.** 0
12. 0 **13.** $\nabla^2\phi$ **14.** $-2\mathbf{a}$

True or False

1. *F* **2.** *T* **3.** *F* **4.** *T* **5.** *T*
6. *F* **7.** *T*

Chapter

10

Integration of Vectors

1 Integration of Vector Functions

We shall define *integration as the reverse process of differentiation.* Let **f** (t) and **F** (t) be two vector functions of the scalar t such that

$$\frac{d}{dt}\,\mathbf{F}(t)=\mathbf{f}(t).$$

Then $\mathbf{F}(t)$ is called the *indefinite integral* of $\mathbf{f}(t)$ with respect to t and symbolically we write

$$\int \mathbf{f}(t)\,dt=\mathbf{F}(t). \qquad \ldots(1)$$

The function $\mathbf{f}(t)$ to be integrated is called the *integrand.*

If **c** is any *arbitrary constant vector* independent of t, then

$$\frac{d}{dt}\{\mathbf{F}(t)+\mathbf{c}\}=\mathbf{f}(t).$$

This is equivalent to

$$\int \mathbf{f}(t)\,dt=\mathbf{F}(t)+\mathbf{c}. \qquad \ldots(2)$$

From (2) it is obvious that the integral $\mathbf{F}(t)$ of $\mathbf{f}(t)$ is indefinite to the extent of an additive arbitrary constant **c** . Therefore $\mathbf{F}(t)$ is called the indefinite integral of $\mathbf{f}(t)$.

The constant vector **c** is called the *constant of integration*. It can be determined if we are given some initial conditions.

If $\frac{d}{dt}\,\mathbf{F}(t)=\mathbf{f}(t)$ for all t in the interval $[a\,,b]$, then the *definite integral* between the limits $t=a$ and $t=b$ can, in such case, be written as

$$\int_a^b \mathbf{f}(t)\,dt=\int_a^b\left\{\frac{d}{dt}\,\mathbf{F}(t)\right\}dt=\big[\mathbf{F}(t)+\mathbf{c}\big]_a^b=\mathbf{F}(b)-\mathbf{F}(a).$$

Theorem: *If* $\mathbf{f}(t)=f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}$, *then*

$$\int \mathbf{f}(t)\,dt=\mathbf{i}\int f_1(t)\,dt+\mathbf{j}\int f_2(t)\,dt+\mathbf{k}\int f_3(t)\,dt.$$

Proof: Let $\frac{d}{dt}\,\mathbf{F}(t)=\mathbf{f}(t)$. ...(1)

Then $\int \mathbf{f}(t)\,dt=\mathbf{F}(t)$. ...(2)

Let $\mathbf{F}(t)=F_1(t)\,\mathbf{i}+F_2(t)\,\mathbf{j}+F_3(t)\,\mathbf{k}$.

Then from (1), we have

$$\frac{d}{dt}\{F_1(t)\,\mathbf{i}+F_2(t)\,\mathbf{j}+F_3(t)\,\mathbf{k}\}=\mathbf{f}(t)$$

or
$$\left\{\frac{d}{dt}F_1(t)\right\}\mathbf{i}+\left\{\frac{d}{dt}F_2(t)\right\}\mathbf{j}+\left\{\frac{d}{dt}F_3(t)\right\}\mathbf{k}=f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}.$$

Equating the coefficients of **i**, **j**, **k**, we get

$$\frac{d}{dt}F_1(t)=f_1(t),\ \frac{d}{dt}F_2(t)=f_2(t),\ \frac{d}{dt}F_3(t)=f_3(t).$$

$$\therefore\quad F_1(t)=\int f_1(t)\,dt,\ F_2(t)=\int f_2(t)\,dt,\ F_3(t)=\int f_3(t)\,dt.$$

$$\therefore\quad \mathbf{F}(t)=\left\{\int f_1(t)\,dt\right\}\mathbf{i}+\left\{\int f_2(t)\,dt\right\}\mathbf{j}+\left\{\int f_3(t)\,dt\right\}\mathbf{k}.$$

So from (2), we get

$$\int \mathbf{f}(t)\,dt=\mathbf{i}\int f_1(t)\,dt+\mathbf{j}\int f_2(t)\,dt+\mathbf{k}\int f_3(t)\,dt.$$

Note: From this theorem we conclude that the definition of the integral of a vector function implies the definition of integrals of three scalar functions which are the components of that vector function. **Thus in order to integrate a vector function we should integrate its components.**

2 Some Standard Results

We have already obtained some standard results for differentiation. With the help of these results we can obtain some standard results for integration.

1. We have $\frac{d}{dt}(\mathbf{r}\bullet\mathbf{s})=\frac{d\mathbf{r}}{dt}\bullet\mathbf{s}+\mathbf{r}\bullet\frac{d\mathbf{s}}{dt}\cdot$

Therefore $\int \left(\frac{d\mathbf{r}}{dt} \bullet \mathbf{s} + \mathbf{r} \bullet \frac{d\mathbf{s}}{dt}\right) dt = \mathbf{r} \bullet \mathbf{s} + c,$

where c is the constant of integration. It should be noted that c is here a scalar quantity since the integrand is also scalar.

2. We have $\frac{d}{dt}(\mathbf{r}^2) = 2\mathbf{r} \bullet \frac{d\mathbf{r}}{dt}\cdot$ Therefore $\int \left(2\,\mathbf{r} \bullet \frac{d\mathbf{r}}{dt}\right) dt = \mathbf{r}^2 + c.$

   Here the constant of integration c is a scalar quantity.

3. We have $\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2 = 2\,\frac{d\mathbf{r}}{dt} \bullet \frac{d^2\mathbf{r}}{dt^2}\cdot$

   Therefore we have $\int \left(2\,\frac{d\mathbf{r}}{dt} \bullet \frac{d^2\mathbf{r}}{dt^2}\right) dt = \left(\frac{d\mathbf{r}}{dt}\right)^2 + c.$

   Here the constant of integration c is a scalar quantity.

   Also $\left(\frac{d\mathbf{r}}{dt}\right)^2 = \frac{d\mathbf{r}}{dt} \bullet \frac{d\mathbf{r}}{dt}\cdot$

4. We have $\frac{d}{dt}\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) = \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} + \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\cdot$

   $\therefore \quad \int \left(\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\right) dt = \mathbf{r} \times \frac{d\mathbf{r}}{dt} + \mathbf{c}.$

   Here the constant of integration **c** is a vector quantity since the integrand $\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}$ is also a vector quantity.

5. If **a** is a constant vector, we have

   $$\frac{d}{dt}(\mathbf{a} \times \mathbf{r}) = \frac{d\mathbf{a}}{dt} \times \mathbf{r} + \mathbf{a} \times \frac{d\mathbf{r}}{dt} = \mathbf{a} \times \frac{d\mathbf{r}}{dt}\cdot$$

   Therefore $\int \left(\mathbf{a} \times \frac{d\mathbf{r}}{dt}\right) dt = \mathbf{a} \times \mathbf{r} + \mathbf{c}.$

   Here the constant of integration **c** is a vector quantity.

6. If $r = |\mathbf{r}|$ and $\hat{\mathbf{r}}$ is a unit vector in the direction of **r**, then

   $$\frac{d}{dt}(\hat{\mathbf{r}}) = \frac{d}{dt}\left(\frac{1}{r}\,\mathbf{r}\right) = \frac{1}{r}\frac{d\,\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\,\mathbf{r}.$$

   Therefore $\int \left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\,\mathbf{r}\right) dt = \hat{\mathbf{r}} + \mathbf{c}.$

7. If c is a constant scalar and **r** a vector function of a scalar t, then obviously

   $$\int c\mathbf{r}\, dt = c \int \mathbf{r}\, dt.$$

8. If **r** and **s** are two vector functions of the scalar t, then obviously

   $$\int (\mathbf{r} + \mathbf{s})\, dt = \int \mathbf{r}\, dt + \int \mathbf{s}\, dt.$$

Illustrative Examples

Example 1: *Evaluate* $\int_1^2 \mathbf{r} \times \dfrac{d^2\mathbf{r}}{dt^2}\, dt$, *where* $\mathbf{r} = 2t^2\ \mathbf{i} + t\ \mathbf{j} - 3t^3\ \mathbf{k}$.

(Bundelkhand 2004; Kumaun 08; Kanpur 13)

Solution: Given $\mathbf{r} = 2t^2\mathbf{i} + t\ \mathbf{j} - 3t^3\ \mathbf{k}$.

$$\therefore \qquad \frac{d\mathbf{r}}{dt} = 4t\ \mathbf{i} + \mathbf{j} - 9t^2\ \mathbf{k} \text{ and } \frac{d^2\mathbf{r}}{dt^2} = 4\mathbf{i} + 0\,\mathbf{j} - 18t\mathbf{k}.$$

$$\therefore \qquad \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = (2t^2\mathbf{i} + t\mathbf{j} - 3t^3\mathbf{k}) \times (4\mathbf{i} + 0\,\mathbf{j} - 18t\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t^2 & t & -3t^3 \\ 4 & 0 & -18t \end{vmatrix}$$

$$= -18t^2\mathbf{i} - (-36t^3 + 12t^3)\ \mathbf{j} - 4t\ \mathbf{k}$$

$$= -18\ t^2\mathbf{i} + 24t^3\ \mathbf{j} - 4t\mathbf{k}.$$

$$\therefore \qquad \int_1^2 \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\, dt = \int_1^2 (-18t^2\mathbf{i} + 24t^3\ \mathbf{j} - 4t\mathbf{k})\ dt$$

$$= -18\mathbf{i}\int_1^2 t^2\, dt + 24\mathbf{j}\int_1^2 t^3\, dt - 4\mathbf{k}\int_1^2 t\ dt$$

$$= -18\mathbf{i}\left[\frac{t^3}{3}\right]_1^2 + 24\mathbf{j}\left[\frac{t^4}{4}\right]_1^2 - 4\mathbf{k}\left[\frac{t^2}{2}\right]_1^2$$

$$= -6\ (8-1)\ \mathbf{i} + 6\ (16-1)\ \mathbf{j} - 2\ (4-1)\ \mathbf{k}$$

$$= -42\mathbf{i} + 90\,\mathbf{j} - 6\mathbf{k}.$$

Example 2: *Find the value of* $\mathbf{r}$ *satisfying the equation* $\dfrac{d^2\mathbf{r}}{dt^2} = t\mathbf{a} + \mathbf{b}$, *where* $\mathbf{a}$ *and* $\mathbf{b}$ *are constant vectors.*

Solution: Integrating the equation $\dfrac{d^2\mathbf{r}}{dt^2} = t\mathbf{a} + \mathbf{b}$, we get $\dfrac{d\mathbf{r}}{dt} = \dfrac{1}{2}t^2\ \mathbf{a} + t\mathbf{b} + \mathbf{c}$, where $\mathbf{c}$ is constant.

Again integrating, we get

$$\mathbf{r} = \frac{1}{6}t^3 + \frac{1}{2}t^2\ \mathbf{b} + t\mathbf{c} + \mathbf{d}\text{, where } \mathbf{d} \text{ is constant.}$$

Example 3: *If* $\mathbf{r}\ (t) = 5t^2\mathbf{i} + t\mathbf{j} - t^3\mathbf{k}$, *prove that*

$$\int_1^2 \left(\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\right) dt = -14\mathbf{i} + 75\mathbf{j} - 15\mathbf{k}.$$

(Kumaun 2000; Meerut 01, 04, 05, 07, 10, 10B; Kanpur 10; Rohilkhand 13)

Solution: We have $\int \left(\mathbf{r} \times \dfrac{d^2 \mathbf{r}}{dt^2} \right) dt = \mathbf{r} \times \dfrac{d\mathbf{r}}{dt} + \mathbf{c}.$

$$\therefore \qquad \int_1^2 \left(\mathbf{r} \times \frac{d^2 \mathbf{r}}{dt^2} \right) dt = \left[\mathbf{r} \times \frac{d\mathbf{r}}{dt} \right]_1^2 \cdot$$

Let us now find $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} \cdot$ We have $\dfrac{d\mathbf{r}}{dt} = 10\, t\ \mathbf{i} + \mathbf{j} - 3t^2\ \mathbf{k}.$

$$\therefore \qquad \mathbf{r} \times \frac{d\mathbf{r}}{dt} = (5t^2\ \mathbf{i} + t\ \mathbf{j} - t^3\ \mathbf{k}) \times (10t\ \mathbf{i} + \mathbf{j} - 3t^2\ \mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix} = -\,2t^3\ \mathbf{i} + 5t^4\ \mathbf{j} - 5t^2\ \mathbf{k}.$$

$$\therefore \qquad \int_1^2 \left(\mathbf{r} \times \frac{d^2 \mathbf{r}}{dt^2} \right) dt = \left[-2t^3\ \mathbf{i} + 5t^4 \mathbf{j} - 5t^2 \mathbf{k} \right]_1^2$$

$$= \left[-\,2t^3 \right]_1^2 \mathbf{i} + \left[5t^4 \right]_1^2 \mathbf{j} - \left[5t^2 \right]_1^2 \mathbf{k}$$

$$= -\,14\mathbf{i} + 75\mathbf{j} - 15\mathbf{k}.$$

Example 4: *Given that* $\mathbf{r}\,(t) = \begin{cases} 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}, & \text{when } \ t = 2 \\ 4\mathbf{i} - 2\ \mathbf{j} + 3\mathbf{k}, & \text{when } \ t = 3, \end{cases}$

show that $\displaystyle\int_2^3 \left(\mathbf{r} \bullet \frac{d\mathbf{r}}{dt} \right) dt = 10.$

(Meerut 2003, 13B; Bundelkhand 08; Kanpur 09, 11; Agra 06; Avadh 10; Purvanchal 13)

Solution: We have $\displaystyle\int \left(\mathbf{r} \bullet \frac{d\mathbf{r}}{dt} \right) dt = \left[\frac{1}{2} \mathbf{r}^2 \right]_2^3 \cdot$

When $\quad t = 3, \mathbf{r} = 4\mathbf{i} - 2\ \mathbf{j} + 3\mathbf{k}.$

$\therefore$ when $\quad t = 3, \mathbf{r}^2 = (4\mathbf{i} - 2\ \mathbf{j} + 3\mathbf{k}) \bullet (4\mathbf{i} - 2\ \mathbf{j} + 3\mathbf{k}) = 16 + 4 + 9 = 29.$

When $\quad t = 2\ , \mathbf{r} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}.$

$\therefore$ when $\quad t = 2\ , \mathbf{r}^2 = 4 + 1 + 4 = 9.$

$$\therefore \qquad \int_2^3 \left(\mathbf{r} \bullet \frac{d\mathbf{r}}{dt} \right) dt = \frac{1}{2}\,[29 - 9] = 10.$$

Example 5: *The acceleration of a particle at any time $t \geq 0$ is given by*

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 12\, cos\ 2t\ \mathbf{i} - 8\, sin\ 2t\ \mathbf{j} + 16\ t\ \mathbf{k}.$$

If the velocity $\mathbf{v}$ *and displacement* $\mathbf{r}$ *are zero at* $t = 0$, *find* $\mathbf{v}$ *and* $\mathbf{r}$ *at any time.*

(Agra 2007)

Solution: We have $\dfrac{d\mathbf{v}}{dt} = 12 \cos 2t\ \mathbf{i} - 8 \sin 2t\ \mathbf{j} + 16t\ \mathbf{k}.$

Integrating, we get

$$\mathbf{v} = \mathbf{i}\int 12\cos 2t\,dt + \mathbf{j}\int -8\sin 2t\,dt + \mathbf{k}\int 16t\,dt$$

or $$\mathbf{v} = 6\sin 2t\,\mathbf{i} + 4\cos 2t\,\mathbf{j} + 8t^2\,\mathbf{k} + \mathbf{c}.$$

When $t = 0$, $\mathbf{v} = \mathbf{0}$.

$\therefore$ $$\mathbf{0} = 0\mathbf{i} + 4\mathbf{j} + 0\mathbf{k} + \mathbf{c} \quad \text{or} \quad \mathbf{c} = -4\mathbf{j}.$$

$\therefore$ $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = 6\sin 2t\,\mathbf{i} + (4\cos 2t - 4)\,\mathbf{j} + 8t^2\,\mathbf{k}.$$

Integrating, we get

$$\mathbf{r} = \mathbf{i}\int 6\sin 2t\,dt + \mathbf{j}\int (4\cos 2t - 4)\,dt + \mathbf{k}\int 8t^2\,dt$$

$$= -3\cos 2t\,\mathbf{i} + (2\sin 2t - 4t)\,\mathbf{j} + \frac{8}{3}t^3\,\mathbf{k} + \mathbf{d},$$

where $\mathbf{d}$ is constant.

When $t = 0$, $\mathbf{r} = \mathbf{0}$.

$\therefore$ $$\mathbf{0} = -3\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} + \mathbf{d}. \qquad \therefore \qquad \mathbf{d} = 3\mathbf{i}.$$

$\therefore$ $$\mathbf{r} = -3\cos 2t\,\mathbf{i} + (2\sin 2t - 4t)\,\mathbf{j} + \frac{8}{3}t^3\,\mathbf{k} + 3\mathbf{i}$$

$$= (3 - 3\cos 2t)\,\mathbf{i} + (2\sin 2t - 4t)\,\mathbf{j} + \frac{8}{3}t^3\,\mathbf{k}.$$

Comprehensive Exercise 1

1. If $\mathbf{f}(t) = (t - t^2)\,\mathbf{i} + 2t^3\,\mathbf{j} - 3\mathbf{k}$, find

   (i) $\int \mathbf{f}(t)\,dt$ (ii) $\int_1^2 \mathbf{f}(t)\,dt.$

2. Evaluate $\int_0^1 (e^t\,\mathbf{i} + e^{-2t}\,\mathbf{j} + t\,\mathbf{k})\,dt$. **(Garhwal 2001, 02)**

3. If $\mathbf{f}(t) = t\,\mathbf{i} + (t^2 - 2t)\,\mathbf{j} + (3t^2 + 3t^3)\,\mathbf{k}$, find $\int_0^1 \mathbf{f}(t)\,dt.$

   (Garhwal 2003; Bundelkhand 07)

4. If $\mathbf{r} = t\,\mathbf{i} - t^2\,\mathbf{j} + (t - 1)\,\mathbf{k}$ and $\mathbf{s} = 2t^2\,\mathbf{i} + 6t\,\mathbf{k}$, evaluate

   (i) $\int_0^2 \mathbf{r} \bullet \mathbf{s}\,dt,$ (ii) $\int_0^2 \mathbf{r} \times \mathbf{s}\,dt$

   (Rohilkhand 2008)

5. (i) Find the value of $\mathbf{r}$ satisfying the equation $\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}$, where $\mathbf{a}$ is a constant vector. Also it is given that when $t = 0$, $\mathbf{r} = \mathbf{0}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{u}$.

(ii) Solve the equation $\dfrac{d^2\mathbf{r}}{dt^2} = \mathbf{a}$ where $\mathbf{a}$ is a constant vector ; given that

$\mathbf{r} = \mathbf{0}$ and $\dfrac{d\mathbf{r}}{dt} = \mathbf{0}$ when $t = 0$.

(Bundelkhand 2008)

6. Find the value of $\mathbf{r}$ satisfying the equation $\dfrac{d^2\mathbf{r}}{dt^2} = 6t\mathbf{i} - 24t^2\mathbf{j} + 4\sin t\,\mathbf{k}$, given that $\mathbf{r} = 2\mathbf{i} + \mathbf{j}$ and $d\mathbf{r}/dt = -\mathbf{i} - 3\mathbf{k}$ at $t = 0$.

(Agra 2001; Meerut 11)

7. The acceleration of a particle at any time t is $e^t\mathbf{i} + e^{2t}\mathbf{j} + \mathbf{k}$.
Find $\mathbf{v}$, given that $\mathbf{v} = \mathbf{i} + \mathbf{j}$ at $t = 0$.

8. Evaluate $\int_1^2 (\mathbf{a} \bullet \mathbf{b} \times \mathbf{c})\, dt$, where
$\mathbf{a} = t\mathbf{i} - 3\mathbf{j} + 2t\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + t\,\mathbf{j} - \mathbf{k}$. **(Meerut 2013)**

9. Integrate $\dfrac{d^2\mathbf{r}}{dt^2} = -n\mathbf{r}^2$.

(Kumaun 2009)

Answers 1

1. (i) $\left(\dfrac{t^2}{2} - \dfrac{t^3}{3}\right)\mathbf{i} + \dfrac{t^4}{2}\mathbf{j} - 3t\,\mathbf{k} + \mathbf{c}$

   (ii) $-\dfrac{5}{6}\mathbf{i} + \dfrac{15}{2}\mathbf{j} - 3\,\mathbf{k}$

2. $(e - 1)\,\mathbf{i} - \dfrac{1}{2}(e^{-2} - 1)\,\mathbf{j} + \dfrac{1}{2}\mathbf{k}$

3. $\dfrac{1}{2}\mathbf{i} - \dfrac{2}{3}\mathbf{j} + \dfrac{7}{4}\mathbf{k}$

4. (i) 12

   (ii) $-24\,\mathbf{i} - \dfrac{40}{3}\mathbf{j} + \dfrac{64}{5}\mathbf{k}$

5. (i) $\dfrac{1}{2}t^2\,\mathbf{a} + t\mathbf{u}$

   (ii) $\dfrac{1}{2}t^2\,\mathbf{a}$

6. $\mathbf{r} = (t^3 - t + 2)\,\mathbf{i} + (1 - 2t^4)\,\mathbf{j} + (t - 4\sin t)\,\mathbf{k}$

7. $e^t\,\mathbf{i} + \dfrac{1}{2}(e^{2t} + 1)\,\mathbf{j} + t\,\mathbf{k}$

8. 0

9. $-n^2\mathbf{r}^2 + c$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If $\mathbf{F}(t) = t\,\mathbf{i} + (t^2 - 2\,t)\,\mathbf{j} + (3t^2 + 3t^3)\,\mathbf{k}$, then the value of $\int_0^1 \mathbf{F}(t)\,dt$ is

(a) $\frac{1}{2}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{7}{4}\mathbf{k}$ (b) $\frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{7}{4}\mathbf{k}$

(c) $-\frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{7}{4}\mathbf{k}$ (d) None of these

(Bundelkhand 2001)

2. If $\mathbf{r} = t\,\mathbf{i} - t^2\,\mathbf{j} + (t-1)\,\mathbf{k}$, and $\mathbf{s} = 2t^2\,\mathbf{i} + 6t\,\mathbf{k}$, then the value of $\int_0^1 \mathbf{r} \bullet \mathbf{s}\,dt$ is

(a) 10 (b) 12

(c) 15 (d) None of these

(Kumaun 2007, 10)

3. The value of $\int_0^1 (e^t\,\mathbf{i} + e^{-2t}\,\mathbf{j} + t\,\mathbf{k})\,dt$ is

(a) $e\,\mathbf{i} - \frac{1}{2}e^{-2}\,\mathbf{j} + \frac{1}{2}\mathbf{k}$ (b) $(e-1)\,\mathbf{i} - \frac{1}{2}(e^{-2}+1)\,\mathbf{j} + \frac{1}{2}\mathbf{k}$

(c) $(e-1)\,\mathbf{i} - \frac{1}{2}(e^{-2}-1)\,\mathbf{j} + \frac{1}{2}\mathbf{k}$ (d) none of these

(Garhwal 2008)

4. The value of $\boldsymbol{r}$ satisfying the equation $\frac{d^2\boldsymbol{r}}{dt^2} = t\mathbf{a} + \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors is

(a) $\frac{t^3}{6} + \frac{t^2}{2}\mathbf{b} + t\mathbf{c} + \mathbf{d}$ (b) $\frac{t^3}{6} - \frac{t^2}{2}\mathbf{b} + t\mathbf{c} + \mathbf{d}$

(c) $\frac{t^3}{6} - \frac{t^2}{2}\mathbf{b} - t\mathbf{c} + \mathbf{d}$ (d) $-\frac{t^3}{6} - \frac{t^2}{2}\mathbf{b} - t\mathbf{c} + \mathbf{d}$

5. The acceleration of a particle at any time $t \geq 0$ is given by
$\mathbf{a} = \frac{d\boldsymbol{v}}{dt} = 12\cos 2t\,\mathbf{i} - 8\sin 2t\,\mathbf{j} + 16t\,\mathbf{k}$, the velocity $\boldsymbol{v}$ at any time is

(a) $6\sin 2t\mathbf{i} - 4\cos 2t\mathbf{j} + 8t^2\,\mathbf{k} + \mathbf{c}$ (b) $6\sin 2t\mathbf{i} + 4\cos 2t\mathbf{j} + 8t^2\,\mathbf{k} + \mathbf{c}$

(c) $6\sin 2t\mathbf{i} - 4\cos 2t\mathbf{j} - 8t^2\,\mathbf{k} + \mathbf{c}$ (d) none of these

6. If $f(t)=(t-t^2)\,\mathbf{i}+2t^3\,\mathbf{j}-3\,\mathbf{k}$, the value of $\int f(t)\,dt$ is

(a) $\left(\frac{t^2}{2}+\frac{t^3}{3}\right)\mathbf{i}+\frac{t^4}{2}\mathbf{j}-3t\mathbf{k}+\mathbf{c}$ (b) $\left(\frac{t^2}{2}-\frac{t^3}{3}\right)\mathbf{i}+\frac{t^4}{2}\mathbf{j}-3t\mathbf{k}+\mathbf{c}$

(c) $\left(\frac{t^2}{2}-\frac{t^3}{3}\right)\mathbf{i}-\frac{t^4}{2}\mathbf{j}-3t\mathbf{k}+\mathbf{c}$ (d) $\left(\frac{t^2}{2}+\frac{t^3}{3}\right)\mathbf{i}-\frac{t^4}{2}\mathbf{j}-3t\mathbf{k}+\mathbf{c}$

7. The acceleration of a particle at any time t is $e^t\,\mathbf{i}+e^{2t}\,\mathbf{j}+\mathbf{k}$, the velocity $\boldsymbol{v}$ at any time is

(a) $e^t\mathbf{i}+2e^{2t}\mathbf{j}+\mathbf{k}+c$ (b) $e^t\mathbf{i}+\frac{1}{2}e^{2t}\mathbf{j}+c$

(c) $e^t\mathbf{i}+\frac{1}{2}e^{2t}\mathbf{j}+t\mathbf{k}+c$ (d) none of these

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. $\int_0^1 [t\,\mathbf{i}+(t^2-2t)\,\mathbf{j}]\,dt=\ldots\ldots$

2. If $\mathbf{F}(t)=3t^2\,\mathbf{i}+t\,\mathbf{j}+2\,\mathbf{k}$ and $\mathbf{G}(t)=6t^2\,\mathbf{i}+(t-1)\,\mathbf{j}+3t\,\mathbf{k}$, then

$$\int_0^1\left(\frac{d\mathbf{F}}{dt}\bullet\mathbf{G}+\mathbf{F}\bullet\frac{d\mathbf{G}}{dt}\right)dt=\ldots\ldots$$

(Meerut 2011)

3. If $\mathbf{r}(t)=5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k}$, then $\int_1^2 \mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\,dt=\ldots\ldots$ **(Kumaun 2013)**

4. If $\mathbf{r}=t^2\,\mathbf{i}+\mathbf{j}-t\,\mathbf{k}$, then $\int_0^1 \mathbf{r}\bullet\frac{d\mathbf{r}}{dt}\,dt=\ldots\ldots$ **(Kumaun 2014)**

True or False

Write 'T' for true and 'F' for false statement.

1. $\int\left(2\,\frac{d\mathbf{r}}{dt}\bullet\frac{d^2\mathbf{r}}{dt^2}\right)dt=\left(\frac{d\mathbf{r}}{dt}\right)^2+c.$

2. The value of $\int_0^1 (e^t\,\mathbf{i}+e^{-2t}\,\mathbf{j}+t\,\mathbf{k})\,dt$ is $(e-1)\,\mathbf{i}-(e^{-2}-1)\,\mathbf{j}+\mathbf{k}$.

Answers

Multiple Choice Questions

1. (b) **2.** (d) **3.** (c) **4.** (a) **5.** (b)
6. (b) **7.** (c)

Fill in the Blank(s)

1. $\frac{1}{2}i - \frac{2}{3}j$

2. 24

3. $-14\,\mathbf{i} + 75\,\mathbf{j} - 15\,\mathbf{k}$

4. 1

True or False

1. *T*

2. *F*

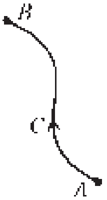
□

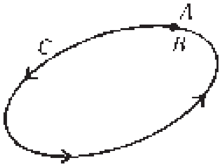
Chapter 11

Line Integrals

1 Some Preliminary Concepts

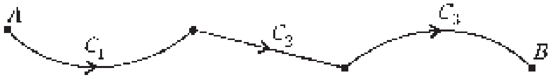
Oriented curve. Suppose C is a curve in space. Let us orient C by taking one of the two directions along C as the *positive direction*; the opposite direction along C is then called the *negative direction* . Suppose A is the initial point and B the terminal point of C under the chosen orientation. In case these two points coincide, the curve C is called a *closed curve*.





Oriented closed curve

Smooth curve. Let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, where $\mathbf{r}(t)$ is the position vector of (x, y, z), be the parametric representation of a curve C joining the points A and B, where $t = t_1$ and $t = t_2$ respectively. We know that $d\mathbf{r} / dt$ is a tangent vector to this curve at the point **r**. Suppose the function **r** (t) is continuous and has a continuous first derivative not equal to zero vector for all values of t under consideration. Then the curve C possesses a unique tangent at each of its points. A curve satisfying these assumptions is called a **smooth curve.**

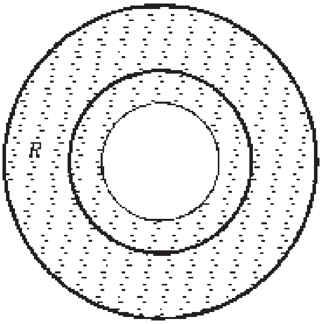


Piecewise Smooth Curve

A curve C is said to be **piecewise smooth** if it is composed of a finite number of smooth curves. The curve C in the adjoining figure is piecewise smooth as it is composed of three smooth curves C_1, C_2 and C_3 . The circle is a smooth closed curve while the curve consisting of the four sides of a rectangle is a piecewise smooth closed curve.

Smooth surface. Suppose S is a surface which has a unique normal at each of its points and the direction of this normal depends continuously on the points of S. Then S is called a **smooth surface.**

If a surface S is not smooth but can be subdivided into a finite number of smooth surfaces, then it is called a **piecewise smooth surface.** The surface of a sphere is smooth while the surface of a cube is piecewise smooth.



Classification of regions. A region R in which every closed curve can be contracted to a point without passing out of the region is called a **simply connected region.** Otherwise the region R is **multiply-connected**. The region interior to a circle is a simply-connected plane region. The region interior to a sphere is a simply-connected region in space. The region between two concentric circles lying in the same plane is a multiply-connected plane region.

If we take a closed curve in this region surrounding the inner circle, then it cannot be contracted to a point without passing out of the region. Therefore the region is not simply-connected. However the region between two concentric spheres is a simply-connected region in space. The region between two infinitely long coaxial cylinders is a multiply-connected region in space.

2 Line Integrals

(Avadh 2014)

Any integral which is to be evaluated along a curve is called a line integral.

Suppose $\mathbf{r}\ (t) = x\ (t)\ \mathbf{i} + y\ (t)\ \mathbf{j} + z\ (t)\ \mathbf{k}$,where $\mathbf{r}\ (t)$ is the position vector of (x, y, z) *i.e.*, $\mathbf{r}\ (t) = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$,defines a piecewise smooth curve joining two points A and B. Let $t = t_1$ at A and $t = t_2$ at B. Suppose $\mathbf{F}(x, y, z) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a vector point function defined and continuous along C. If s denotes the arc length of the curve C, then $\dfrac{d\mathbf{r}}{ds} = \mathbf{t}$ is a unit vector along the tangent to the curve C at the point $\mathbf{r}$. The component of the vector $\mathbf{F}$ along this tangent is $\mathbf{F} \bullet \dfrac{d\mathbf{r}}{ds}$. The integral of $\mathbf{F} \bullet \dfrac{d\mathbf{r}}{ds}$ along C from A to B written as

$$\int_A^B \left[\mathbf{F} \bullet \frac{d\mathbf{r}}{ds}\right] ds = \int_A^B \mathbf{F} \bullet d\mathbf{r} = \int_C \mathbf{F} \bullet d\mathbf{r}$$

is an example of a *line integral*. It is called *the tangent line integral* of $\mathbf{F}$ along C.

Since $\mathbf{r} = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$, therefore, $d\mathbf{r} = dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k}$.

$$\therefore \quad \mathbf{F} \bullet d\mathbf{r} = (F_1\mathbf{i} + F_2\ \mathbf{j} + F_3\mathbf{k}) \bullet (dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k})$$
$$= F_1\ dx + F_2\ dy + F_3\ dz.$$

Therefore in components form the above line integral is written as

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (F_1\ dx + F_2\ dy + F_3\ dz).$$

The parametric equations of the curve C are $x = x(t)$, $y = y(t)$ and $z = z(t)$. Therefore we may write

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_{t_1}^{t_2} \left[F_1\ \frac{dx}{dt} + F_2\ \frac{dy}{dt} + F_3\ \frac{dz}{dt}\right] dt.$$

Circulation: *If C is a simple closed curve (i.e. a curve which does not intersect itself anywhere), then the tangent line integral of* $\mathbf{F}$ *around C is called the circulation of* $\mathbf{F}$ *about C. It is often denoted by*

$$\oint \mathbf{F} \bullet d\mathbf{r} = \oint (F_1\ dx + F_2\ dy + F_3\ dz).$$

Work done by a Force. Suppose a force $\mathbf{F}$ acts upon a particle. Let the particle be displaced along a given path C in space. If $\mathbf{r}$ denotes the position vector of a point on C, then $\dfrac{d\mathbf{r}}{ds}$ is a unit vector along the tangent to C at the point $\mathbf{r}$ in the direction of s increasing. The component of force $\mathbf{F}$ along tangent to C is $\mathbf{F} \bullet \dfrac{d\mathbf{r}}{ds}$. Therefore the work done by $\mathbf{F}$ during a small displacement ds of the particle along C is $\left[\mathbf{F} \bullet \dfrac{d\mathbf{r}}{ds}\right] ds$ *i.e.*, $\mathbf{F} \bullet d\mathbf{r}$. The total work W done by $\mathbf{F}$ in this displacement along C, is given by the line integral

$$W = \int_C \mathbf{F} \bullet d\mathbf{r},$$

the integration being taken in the sense of the displacement.

Illustrative Examples

Example 1: *Evaluate* $\int_C \mathbf{F} \bullet d\mathbf{r}$, *where* $\mathbf{F} = x^2\,\mathbf{i} + y^3\,\mathbf{j}$ *and curve C is the arc of the parabola* $y = x^2$ *in the x-y plane from (0, 0) to (1, 1).*

Solution: We shall illustrate two methods for the solution of such a problem.

Method 1. The curve C is the parabola $y = x^2$ from (0, 0) to (1, 1).

Let $x = t$; then $y = t^2$. If $\mathbf{r}$ is the position vector of any point (x, y) on C, then

$$\mathbf{r}\,(t) = x\mathbf{i} + y\mathbf{j} = t\mathbf{i} + t^2\,\mathbf{j}. \qquad \therefore \qquad \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j}.$$

Also in terms of t, $\mathbf{F} = t^2\,\mathbf{i} + t^6\,\mathbf{j}$.

At the point (0, 0), $t = x = 0$. At the point (1, 1), $t = 1$.

$$\therefore \qquad \int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \left(\mathbf{F} \bullet \frac{d\mathbf{r}}{dt}\right) dt = \int_0^1 (t^2\ \mathbf{i} + t^6\ \mathbf{j}) \bullet (\mathbf{i} + 2t\ \mathbf{j})\ dt$$

$$= \int_0^1 t^2 + 2t^7)\ dt = \left[\frac{t^3}{3} + \frac{2t^8}{8}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

Method 2: In the xy-plane we have $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

$$\therefore \qquad d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}.$$

Therefore, $\mathbf{F} \bullet d\mathbf{r} = (x^2\,\mathbf{i} + y^3\,\mathbf{j}) \bullet (dx\ \mathbf{i} + dy\ \mathbf{j}) = x^2\,dx + y^3\,dy$.

$$\therefore \qquad \int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (x^2\,dx + y^3\,dy).$$

Now along the curve C, $y = x^2$. Therefore $dy = 2x\ dx$.

$$\therefore \qquad \int_C \mathbf{F} \bullet d\mathbf{r} = \int_{x=0}^1 [x^2\ dx + x^6\ (2x)\ dx]$$

$$= \int_0^1 (x^2 + 2x^7)\ dx = \left[\frac{x^3}{3} + \frac{2x^8}{8}\right]_0^1 = \frac{7}{12}.$$

Example 2: *If* $\mathbf{F} = 3xy\ \mathbf{i} - y^2\ \mathbf{j}$, *evaluate* $\int_C \mathbf{F} \bullet d\mathbf{r}$, *where C is the curve in the xy-plane,* $y = 2x^2$, *from* (0, 0) *to* (1, 2). **(Garhwal 2001, 02; Kumaun 07; Rohilkhand 12)**

Solution: The parametric equations of the parabola $y = 2x^2$ can be taken as

$$x = t,\ y = 2t^2.$$

At the point (0, 0), $x = 0$ and so $t = 0$. Again at the point (1, 2), $x = 1$ and so $t = 1$.

Now $$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (3xy\mathbf{i} - y^2\,\mathbf{j}) \bullet (dx\ \mathbf{i} + dy\ \mathbf{j})$$

$$[\because \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j}, \text{ so that } d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}]$$

$$= \int_C (3xy\, dx - y^2\, dy) = \int_{t=0}^{1} \left(3xy \frac{dx}{dt} - y^2 \frac{dy}{dt}\right) dt$$

$$= \int_0^1 (3 \,.\, t \,.\, 2\, t^2 \,.\, 1 - 4\, t^4 \,.\, 4t)\, dt$$

$$[\because \quad x = t, \;\; y = 2t^2 \;\; \text{so that} \;\; dx/dt = 1 \;\; \text{and} \;\; dy/dt = 4t]$$

$$= \int_0^1 (6t^3 - 16t^5)\, dt = \left[6 \,.\, \frac{t^4}{4} - 16 \,.\, \frac{t^6}{6}\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}.$$

Example 3: *Evaluate* $\int \mathbf{F} \bullet d\mathbf{r}$ *along the curve* $x^2 + y^2 = 1,\; z = 1$ *in the positive direction from* $(0, 1, 1)$ *to* $(1, 0, 1)$ *where* $\mathbf{F} = (2x + yz)\,\mathbf{i} + xz\mathbf{j} + (xy + 2z)\,\mathbf{k}$.

Solution: Let the given curve be denoted by C and let A and B be points $(0, 1, 1)$ and $(1, 0, 1)$ respectively.

Along the given curve C, we have $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\therefore \qquad d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}.$$

$$\therefore \qquad \int_C \mathbf{F} \bullet d\mathbf{r} = \int_A^B [(2x + yz)\,\mathbf{i} + xz\,\mathbf{j} + (xy + 2z)\,\mathbf{k}] \bullet (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_A^B [(2x + yz)\, dx + xz\, dy + (xy + 2z)\, dz]. \qquad \ldots(1)$$

In moving from A to B, x varies from 0 to 1, y varies from 1 to 0 and z remains constant. We have $z = 1$ and so $dz = 0$.

Hence from (1)

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^1 (2x + y)\, dx + \int_1^0 x\, dy + 0$$

$$= \int_0^1 [2x + \sqrt{(1 - x^2)}]\, dx - \int_0^1 \sqrt{(1 - y^2)}\, dy = [x^2]_0^1 = 1,$$

the last two integrals cancel by a property of definite integrals.

Example 4: *Evaluate* $\int (x\, dy - y\, dx)$ *around the circle* $x^2 + y^2 = 1$. **(Meerut 2002)**

Solution: Let C denote the circle $x^2 + y^2 = 1$. The parametric equations of this circle are $x = \cos t,\; y = \sin t$.

To integrate around the circle C we should vary t from 0 to 2π.

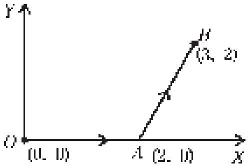
$$\therefore \qquad \oint_C (x\, dy - y\, dx) = \int_0^{2\pi} \left(x \frac{dy}{dt} - y \frac{dx}{dt}\right) dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt = \int_0^{2\pi} dt = 2\pi.$$

Example 5: *If* $\mathbf{F} = (2x + y)\,\mathbf{i} + (3y - x)\,\mathbf{j}$, *evaluate* $\int_C \mathbf{F} \bullet d\mathbf{r}$ *where* C *is the curve in the xy-plane consisting of the straight lines from* $(0, 0)$ *to* $(2, 0)$ *and then to* $(3, 2)$.

(Agra 2007; Meerut 11; Purvanchal 14)

Solution: The path of integration C has been shown in the figure. It consists of the straight lines OA and AB.



We have $\int_C \mathbf{F} \bullet d\mathbf{r}$

$$= \int_C [(2x + y)\,\mathbf{i} + (3y - x)\,\mathbf{j}] \bullet (dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C [(2x + y)\,dx + (3y - x)\,dy].$$

Now along the straight line OA, $y = 0$, $dy = 0$ and x varies from 0 to 2. The equation of the straight line AB is

$$y - 0 = \frac{2-0}{3-2}(x - 2) \quad i.e., \quad y = 2x - 4.$$

$\therefore$ along AB, $y = 2x - 4$, $dy = 2\,dx$ and x varies from 2 to 3.

$$\therefore \quad \int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^2 [(2x + 0)\,dx + 0] + \int_2^3 [(2x + 2x - 4)\,dx + (6x - 12 - x)\,2\,dx]$$

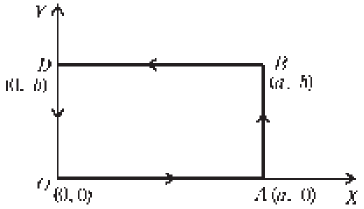
$$= \left[\,x^2\,\right]_0^2 + \int_2^3 (14x - 28)\,dx$$

$$= 4 + 14\int_2^3 (x - 2)\,dx = 4 + 14\left[\frac{(x-2)^2}{2}\right]_2^3 = 4 + 7 = 11.$$

Example 6: *Evaluate* $\int_C \mathbf{F} \bullet d\mathbf{r}$ *where* $\mathbf{F} = (x^2 + y^2)\,\mathbf{i} - 2xy\mathbf{j}$, *curve C is the rectangle in the x y-plane bounded by* $y = 0$, $x = a$, $y = b$, $x = 0$.

(Meerut 2000, 06B, 07B, 09B, 12, 13; Kanpur 10; Bundelkhand 09; Purvanchal 09)

Solution: In the xy-plane $z = 0$.



Therefore $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$.

The path of integration C has been shown in the figure. It consists of the straight lines OA, AB, BD and DO.

We have

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C [(x^2 + y^2)\,\mathbf{i} - 2xy\,\mathbf{j}] \bullet (dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C [(x^2 + y^2)\,dx - 2xy\,dy].$$

Now on OA, $y = 0$, $dy = 0$ and x varies from 0 to a; on AB, $x = a$, $dx = 0$ and y varies from 0 to b; on BD, $y = b$, $dy = 0$ and x varies from a to 0; on DO, $x = 0$, $dx = 0$ and y varies from b to 0.

$$\therefore \quad \int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^a x^2\,dx - \int_0^b 2ay\,dy + \int_a^0 (x^2 + b^2)\,dx + \int_b^0 0\,dy$$

$$= \left[\frac{x^3}{3} \right]_0^a - 2a \left[\frac{y^2}{2} \right]_0^b + \left[\frac{x^3}{3} + b^2 x \right]_a^0 + 0 = -\,2ab^2 .$$

Example 7: *Find the total work done in moving a particle in a force field given by*

$$\mathbf{F} = 3xy\mathbf{i} - 5z\mathbf{j} + 10x\mathbf{k}$$

along the curve $x = t^2 + 1,\ \ y = 2t^2,\ \ z = t^3$ *from* $t = 1$ *to* $t = 2.$

Solution: Let C denote the arc of the given curve from $t = 1$ to $t = 2$. Then the total work done

$$= \int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (3xy\mathbf{i} - 5z\ \mathbf{j} + 10x\ \mathbf{k}) \bullet (dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k})$$

$$= \int_C (3xy\ dx - 5z\ dy + 10x\ dz)$$

$$= \int_1^2 \left(3xy\ \frac{dx}{dt} - 5z\ \frac{dy}{dt} + 10x\ \frac{dz}{dt} \right) dt$$

$$= \int_1^2 [3\,(t^2 + 1)\,(2t)^2\,(2t) - (5t^3)(4t) + 10\,(t^2 + 1)(3t^2)]\ dt$$

$$= \int_1^2 (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)\ dt$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\ dt = 303.$$

Example 8: *Evaluate* $\int_C \mathbf{F} \bullet d\mathbf{r}$, *where* $\mathbf{F} = yz\ \mathbf{i} + zx\ \mathbf{j} + xy\ \mathbf{k}$ *and* C *is the portion of the curve* $\mathbf{r} = a \cos t\ \mathbf{i} + b \sin t\ \mathbf{j} + ct\ \mathbf{k}$, *from* $t = 0$ *to* $t = \pi / 2.$ **(Avadh 2010)**

Solution: Along the curve C,

$$\mathbf{r} = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k} = a \cos t\ \mathbf{i} + b \sin t\ \mathbf{j} + ct\ \mathbf{k}.$$

$\therefore$ $\quad x = a \cos t,\ \ y = b \sin t,\ \ z = ct.$

Now

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (\,yz\ \mathbf{i} + z\,x\ \mathbf{j} + xy\ \mathbf{k}) \bullet (dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k})$$

$$= \int_C (\,yz\ dx + z\,x\ dy + xy\ dz) = \int_C d\,(xyz)$$

$$= \left[xyz \right]_{t=0}^{t=\pi/2} = \left[(a \cos t)\,.\,(b \sin t)\,.\,(ct) \right]_0^{\pi/2}$$

$$= abc\,\left[t \cos t \sin t \right]_0^{\pi/2} = abc\ (0 - 0) = 0.$$

Comprehensive Exercise 1

1. Find $\int_C \mathbf{t} \bullet d\mathbf{r}$ where t is the unit tangent vector and C is the unit circle, in xy-plane, with centre at the origin. **(Bundelkhand 2008)**

2. (i) Integrate the function $\mathbf{F} = x^2\mathbf{i} - xy\mathbf{j}$ from the point $(0, 0)$ to $(1,\ 1)$ along parabola $y^2 = x.$

(ii) Evaluate $\int_C xy^3 \, ds$, where C is the segment of the line $y = 2x$ in the xy-plane from $(-1, -2)$ to $(1, 2)$.

3. Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$, where $\mathbf{F} = \mathbf{i} \cos y - \mathbf{j}\, x \sin y$ and C is the curve $y = \sqrt{(1 - x^2)}$ in the xy-plane from $(1, 0)$ to $(0, 1)$. **(Agra 2001)**

4. (i) Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F}$ is $x^2 y^2 \mathbf{i} + y\mathbf{j}$ and C is $y^2 = 4x$ in the xy-plane from $(0, 0)$ to $(4, 4)$.

(Garhwal 2003; Meerut 04, 06; Kanpur 09, 11, 12; Rohilkhand 11)

(ii) Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$, where $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the curve $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, t varying from -1 to $+1$.

(iii) Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$, where $\mathbf{F} = (2x + y)\,\mathbf{i} + (3y - x)\,\mathbf{j} + yzx\,\mathbf{k}$ and C is the curve $x = 2t^2$, $y = t$, $z = t^3$ from $t = 0$ to $t = 1$. **(Kumaun 2011)**

5. Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ where

$\mathbf{F} = c\,[-3a \sin^2 t \cos t\, \mathbf{i} + a\,(2 \sin t - 3 \sin^3 t)\, \mathbf{j} + b \sin 2t\, \mathbf{k}]$

and C is given by $\mathbf{r} = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j} + bt\, \mathbf{k}$ from $t = \pi/4$ to $\pi/2$.

6. Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ and C is the arc of the curve $\mathbf{r} = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + t\, \mathbf{k}$ from $t = 0$ to $t = 2\pi$.

7. Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F} = xy\,\mathbf{i} + (x^2 + y^2)\,\mathbf{j}$ and C is the x-axis from $x = 2$ to $x = 4$ and the straight line $x = 4$ from $y = 0$ to $y = 12$.

8. Find the work done in moving a particle in a force field $\mathbf{F} = 3x^2\,\mathbf{i} + (2xz - y)\,\mathbf{j} + z\,\mathbf{k}$ along the line joining $(0, 0, 0)$ to $(2, 1, 3)$.

9. Calculate $\int_C [(x^2 + y^2)\,\mathbf{i} + (x^2 - y^2)\,\mathbf{j}] \bullet d\mathbf{r}$ where C is the curve :

(i) $y^2 = x$ joining $(0, 0)$ to $(1, 1)$. **(Meerut 2005B)**

(ii) $x^7 = y$ joining $(0, 0)$ to $(1, 1)$.

(iii) consisting of two straight lines joining $(0, 0)$ to $(1, 0)$ and $(1, 0)$ to $(1, 1)$.

(iv) consisting of three straight lines joining $(0, 0)$ to $(2, -2)$, $(2, -2)$ to $(0, -1)$ and $(0, -1)$ to $(1, 1)$.

10. Find the circulation of **F** round the curve C, where $\mathbf{F} = e^x \sin y\, \mathbf{i} + e^x \cos y\, \mathbf{j}$ and C is the rectangle whose vertices are $(0, 0), (1, 0), (1, \frac{1}{2}\pi), (0, \frac{1}{2}\pi)$.

11. Find the circulation of **F** round the curve C, where $\mathbf{F} = (x - y)\,\mathbf{i} + (x + y)\,\mathbf{j}$ and C is the circle $x^2 + y^2 = 4$, $z = 0$.

12. (i) If $\mathbf{F} = (2x^2 + y^2)\,\mathbf{i} + (3y - 4x)\,\mathbf{j}$, evaluate $\int \mathbf{F} \bullet d\mathbf{r}$ around the triangle ABC whose vertices are $A\,(0, 0)$, $B\,(2, 0)$ and $C\,(2, 1)$.

(Kumaun 2008)

(ii) If $\mathbf{F} = (3x^2 + 6y)\,\mathbf{i} - 14\,yz\,\mathbf{j} + 20\,xz^2\,\mathbf{k}$, then evaluate $\int \mathbf{F} \bullet d\mathbf{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve $x = t,\ y = t^2,\ z = t^3$.

(Kumaun 2012)

Answers 1

1. 2π **2.** (i) $\frac{1}{12}$ (ii) $\frac{16}{\sqrt{5}}$ **3.** -1

4. (i) 264 (ii) $\frac{10}{7}$ (iii) $\frac{277}{42}$ **5.** $\frac{1}{2}c\,(a^2 + b^2)$

6. $2\pi + \pi = 3\pi$ **7.** 768 **8.** 16

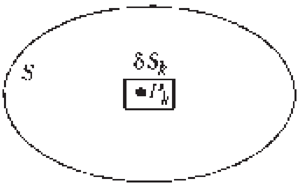
9. (i) $\frac{7}{10}$ (ii) $\frac{38}{45}$ (iii) 1 (iv) $-\frac{7}{3}$ **10.** 0

11. 8π **12.** (i) $-14/3$ (ii) 5

3 Surface Integrals

(Avadh 2014)

Any integral which is to be evaluated over a surface is called a surface integral.



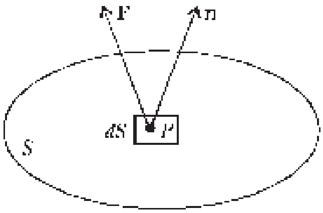
Suppose S is a surface of finite area. Suppose $f\,(x, y, z)$ is a single valued function of position defined over S. Subdivide the area S into n elements of areas $\delta S_1, \delta S_2, \ldots, \delta S_n$. In each part δS_k we choose an arbitrary point P_k whose coordinates are (x_k, y_k, z_k).

We define $f\,(P_k) = f\,(x_k, y_k, z_k)$.

Form the sum $\sum_{k=1}^{n} f\,(P_k)\,\delta S_k$.

Now take the limit of this sum as $n \to \infty$ in such a way that the largest of the areas δS_k approaches zero. This limit if it exists, is called the *surface integral* of $f\,(x, y, z)$ over S and is denoted by $\iint_S f\,(x, y, z)\,dS$.

It can be shown that if the surface S is piecewise smooth and the function $f\,(x, y, z)$ is continuous over S, then the above limit exists *i.e.*, is independent of the choice of sub-divisions and points P_k.



Flux: Suppose S is a piecewise smooth surface and $\mathbf{F}(x, y, z)$ is a vector function of position defined and continuous over S. Let P be any point on the surface S and let $\mathbf{n}$ be the unit vector at P in the direction of **outward drawn normal** to the surface S at P. Then $\mathbf{F} \bullet \mathbf{n}$ is the normal component of $\mathbf{F}$ at P. The integral of $\mathbf{F} \bullet \mathbf{n}$ over S is $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$.

It is called the *flux* of $\mathbf{F}$ over S.

Let us associate with the differential of surface area dS a vector $d\mathbf{S}$ (called *vector area*) whose magnitude is dS and whose direction is that of $\mathbf{n}$. Then $d\mathbf{S} = \mathbf{n}\, dS$. Therefore we can write $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \iint_S \mathbf{F} \bullet d\mathbf{S}$.

Suppose the outward drawn normal to the surface S at P makes angles α, β, γ with the positive directions of x, y and z axes respectively. If l, m, n are the direction cosines of the outward drawn normal, then

$$l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma.$$

Also $\mathbf{n} = \cos\alpha\, \mathbf{i} + \cos\beta\, \mathbf{j} + \cos\gamma\, \mathbf{k} = l\, \mathbf{i} + m\, \mathbf{j} + n\, \mathbf{k}$.

Let $\mathbf{F}(x, y, z) = F_1\, \mathbf{i} + F_2\, \mathbf{j} + F_3\, \mathbf{k}$. Then

$$\mathbf{F} \bullet \mathbf{n} = F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma = F_1 l + F_2 m + F_3 n.$$

Therefore, we can write

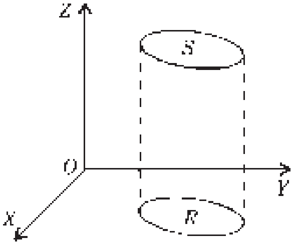
$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dS$$

$$= \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy),$$

if we define $\iint_S F_1 \cos\alpha\, dS = \iint_S F_1\, dy\, dz$,

$$\iint_S F_2 \cos\beta\, dS = \iint_S F_2\, dz\, dx, \qquad \iint_S F_3 \cos\gamma\, dS = \iint_S F_3\, dx\, dy.$$

Note 1: Other examples of surface integrals are $\iint_S f\, \mathbf{n}\, dS$, $\iint_S \mathbf{F} \times d\mathbf{S}$ where $f(x, y, z)$ is a scalar function of position.



Note 2: **Important.** In order to evaluate surface integrals it is convenient to express them as double integrals taken over the orthogonal projection of the surface S on one of the coordinate planes. But this is possible only if any line perpendicular to the co-ordinate plane chosen meets the surface S in no more than one point. If the surface S does not satisfy this condition, then it can be sub-divided into surfaces which do satisfy this condition.

Suppose the surface S is such that any line perpendicular to the xy-plane meets S in no more than one point. Then the equation of the surface S can be written in the form $z = h(x, y)$.

Let R be the orthogonal projection of S on the $x\,y$-plane. If γ is the acute angle which the undirected normal $\mathbf{n}$ at $P(x, y, z)$ to the surface S makes with z-axis, then it can be shown that $\cos\gamma\, dS = dx\, dy$,
where dS is the small element of area of surface S at the point P.

Therefore $dS = \dfrac{dx\,dy}{\cos\gamma} = \dfrac{dx\,dy}{|\mathbf{n}\bullet\mathbf{k}|}$, where $\mathbf{k}$ is the unit vector along z-axis.

Hence $$\iint_S \mathbf{F}\bullet\mathbf{n}\, dS = \iint_R \mathbf{F}\bullet\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\bullet\mathbf{k}|}.$$

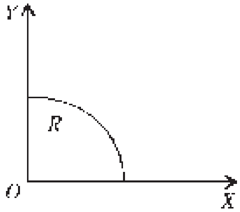
Thus the surface integral on S can be evaluated with the help of a double integral integrated over R.

Illustrative Examples

Example 9: *Evaluate* $\iint_S \mathbf{F}\bullet\mathbf{n}\, dS$, *where* $\mathbf{F} = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$ *and* S *is that part of the surface of the sphere* $x^2 + y^2 + z^2 = 1$ *which lies in the first octant.*

(Garhwal 2003; kumaun 14))

Solution: A vector normal to the surface S is given by
$$\nabla(x^2 + y^2 + z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}.$$



Therefore $\mathbf{n}$ = a unit normal to any point (x, y, z) of S
$$= \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{(4x^2 + 4y^2 + 4z^2)}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$
since $x^2 + y^2 + z^2 = 1$ on the surface S.

We have $$\iint_S \mathbf{F}\bullet\mathbf{n}\, dS = \iint_R \mathbf{F}\bullet\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\bullet\mathbf{k}|},$$
where R is the projection of S on the xy-plane.

The region R is bounded by x-axis, y-axis and the circle $x^2 + y^2 = 1,\ z = 0$.

We have $\mathbf{F}\bullet\mathbf{n} = (yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k})\bullet(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = 3xyz.$

Also $\mathbf{n}\bullet\mathbf{k} = (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\bullet\mathbf{k} = z.\ \therefore\ |\mathbf{n}\bullet\mathbf{k}| = z.$

Hence $$\iint_S \mathbf{F}\bullet\mathbf{n}\, dS = \iint_R \frac{3xyz}{z}\, dx\, dy = 3\iint_R xy\, dx\, dy$$
$$= 3\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} (r\cos\theta)(r\sin\theta)\, r\, d\theta\, dr, \text{ on changing to polars}$$
$$= 3\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 \cos\theta\sin\theta\, d\theta = \frac{3}{4}\left(\frac{1}{2}\right) = \frac{3}{8}.$$

Example 10: *Evaluate* $\iint_S \mathbf{F}\bullet\mathbf{n}\,dS$, *where* $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} - 3y^2 z\mathbf{k}$ *and* S *is the surface of the cylinder* $x^2 + y^2 = 16$ *included in the first octant between* $z = 0$ *and* $z = 5$.

Solution: A vector normal to the surface S is given by

$$\nabla(x^2 + y^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j}.$$

Therefore $\quad \mathbf{n}$ = a unit normal to any point of S

$$= \frac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{(4x^2 + 4y^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{4},$$

since $x^2 + y^2 = 16$, on the surface S.

We have $\iint_S \mathbf{F}\bullet\mathbf{n}\,dS = \iint_R \mathbf{F}\bullet\mathbf{n}\,\dfrac{dx\,dz}{|\mathbf{n}\bullet\mathbf{j}|}$, where R is the projection of S on the x-z plane. It should be noted that in this case we cannot take the projection of S on the x-y plane as the surface S is perpendicular to the x-y plane.

Now
$$\mathbf{F}\bullet\mathbf{n} = (z\,\mathbf{i} + x\,\mathbf{j} - 3y^2 z\,\mathbf{k})\bullet\left(\frac{x\,\mathbf{i} + y\,\mathbf{j}}{4}\right) = \frac{1}{4}(xz + xy),$$

$$\mathbf{n}\bullet\mathbf{j} = \left(\frac{x\,\mathbf{i} + y\,\mathbf{j}}{4}\right)\bullet\mathbf{j} = \frac{y}{4}.$$

Therefore the required surface integral is

$$= \iint_R \frac{xz + xy}{4}\,\frac{dx\,dz}{y/4}$$

$$= \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{(16 - x^2)}} + x\right)dx\,dz, \text{ since } y = \sqrt{(16 - x^2)} \text{ on } S$$

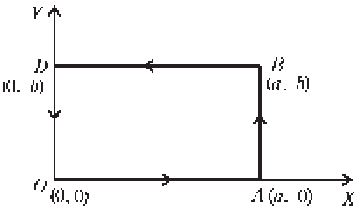
$$= \int_0^5 (4z + 8)\,dz = 90.$$

4 Volume Integrals

(Avadh 2014)

Suppose V is a volume bounded by a surface S. Suppose $f(x, y, z)$ is a single valued function of position defined over V. Subdivide the volume V into n elements of volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. In each part δV_k we choose an arbitrary point P_k whose co-ordinates are (x_k, y_k, z_k). We define

$$f(P_k) = f(x_k, y_k, z_k).$$



Form the sum

$$\sum_{k=1}^{n} f(P_k)\,\delta V_k\,.$$

Now take the limit of this sum as $n \to \infty$ in such a way that the largest of the volumes δV_k approaches zero. This limit, if it exists, is called the volume integral of $f(x, y, z)$ over V and is denoted by $\iiint_V f(x, y, z)\, dV$.

It can be shown that if the surface is piecewise smooth and the function $f(x, y, z)$ is continuous over V, then the above limit exists *i.e.*, is independent of the choice of sub-divisions and points P_k.

If we subdivide the volume V into small cuboids by drawing lines parallel to the three co-ordinates axes, then $dV = dx\, dy\, dz$ and the above volume integral becomes $\iiint_V f(x, y, z)\, dx\, dy\, dz$.

If $\mathbf{F}(x, y, z)$ is a vector function, then $\iiint_V \mathbf{F}\, dV$

is also an example of a volume integral.

Example 11: *If* $\mathbf{F} = (2x^2 - 3z)\,\mathbf{i} - 2xy\,\mathbf{j} - 4x\,\mathbf{k}$, *then evaluate* $\iiint_V \nabla \bullet \mathbf{F}\, dV$ *where* V *is the closed region bounded by the planes* $x = 0,\ y = 0,\ z = 0$ *and* $2x + 2y + z = 4$. *Also Evaluate* $\iiint_V \nabla \times \mathbf{F}\, dV$.

Solution: We have $\mathbf{F} = (2x^2 - 3z)\,\mathbf{i} - 2xy\,\mathbf{j} - 4x\,\mathbf{k}$.

$$\therefore \qquad \nabla \bullet \mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet [(2x^2 - 3z)\,\mathbf{i} - 2xy\,\mathbf{j} - 4x\,\mathbf{k}]$$

$$= \frac{\partial}{\partial x}(2x^2 - 3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x) = 4x - 2x = 2x.$$

$$\therefore \qquad \iiint_V \nabla \bullet \mathbf{F}\, dV = \iiint_V 2x\, dx\, dy\, dz \qquad [\because \ dV = dx\, dy\, dz]$$

$$= 2\int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} x\ dx\, dy\, dz.$$

[Note that we have taken a thin column parallel to z-axis as the elementary volume. It cuts the boundary at $z = 0$ and $z = 4 - 2x - 2y$. Also the projection of the plane $2x + 2y + z = 4$ on the xy-plane is bounded by the axes $y = 0$, $x = 0$ and the line $x + y = 2$. Hence the limits for y are from 0 to $2 - x$ and those for x are from 0 to 2]

$$\therefore \qquad \iiint_V \nabla \bullet \mathbf{F}\, dV = 2\int_{x=0}^{2} \int_{y=0}^{2-x} x \Big[\, z \,\Big]_{z=0}^{4-2x-2y} dx\, dy$$

$$= 2\int_{x=0}^{2} \int_{y=0}^{2-x} x(4 - 2x - 2y)\, dx\, dy$$

$$= 2\int_{x=0}^{2} \Big[4xy - 2x^2 y - xy^2\Big]_{y=0}^{2-x} dx$$

$$= 2\int_0^2 [4x(2 - x) - 2x^2(2 - x) - x(2 - x)^2]\, dx$$

$$= 2\int_0^2 [x^3 - 4x^2 + 4x]\, dx, \text{ on simplifying}$$

$$= 2\left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\right]_0^2 = 2\left[4 - \frac{32}{3} + 8\right] = \frac{8}{3}.$$

Second part: We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-4x) - \frac{\partial}{\partial z}(-2xy)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-4x) - \frac{\partial}{\partial z}(2x^2 - 3z)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(2x^2 - 3z)\right]\mathbf{k}$$

$$= 0\,\mathbf{i} - (-4+3)\,\mathbf{j} + (-2y)\,\mathbf{k} = \mathbf{j} - 2y\,\mathbf{k}.$$

$$\therefore \quad \iiint_V \nabla \times \mathbf{F}\, dV = \iiint_V (\mathbf{j} - 2y\,\mathbf{k})\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} (\mathbf{j} - 2y\,\mathbf{k})\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} (\mathbf{j} - 2y\,\mathbf{k})(4 - 2x - 2y)\, dx\, dy$$

$$= \int_{x=0}^{2} \left[\mathbf{j}(4y - 2xy - y^2) - 2\,\mathbf{k}\left(2y^2 - xy^2 - \frac{2}{3}y^3\right)\right]_{y=0}^{2-x} dx$$

$$= \int_{x=0}^{2} \Big[\mathbf{j}(2-x)(4 - 2x - 2 + x)$$

$$\left. - 2\,\mathbf{k}(2-x)^2\left\{2 - x - \frac{2}{3}(2-x)\right\}\right] dx$$

$$= \int_0^2 \left[(2-x)^2\,\mathbf{j} - \frac{2}{3}(2-x)^3\,\mathbf{k}\right] dx$$

$$= \int_0^2 \left[(x-2)^2\,\mathbf{j} + \frac{2}{3}(x-2)^3\,\mathbf{k}\right] dx$$

$$= \left[\frac{(x-2)^3}{3}\right]_0^2 \mathbf{j} + \left[\frac{2}{3}\,\frac{(x-2)^4}{4}\right]_0^2 \mathbf{k}$$

$$= \frac{8}{3}\,\mathbf{j} - \frac{8}{3}\,\mathbf{k} = \frac{8}{3}(\mathbf{j} - \mathbf{k}).$$

Comprehensive Exercise 2

1. (i) Evaluate $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$, where $\mathbf{F} = 18z\,\mathbf{i} - 12\,\mathbf{j} + 3y\,\mathbf{k}$ and S is the surface of the plane $2x + 3y + 6z = 12$ in the first octant.

(Bundelkhand 2005)

(ii) Evaluate $\iiint_V \phi \, dV$, where $\phi = 45\, x^2 y$ and V is the closed region bounded by the planes $4x + 2y + z = 8$, $x = 0$, $y = 0, z = 0$.

2. Evaluate $\iint_S \mathbf{F} \bullet \mathbf{n} \, dS$, where $\mathbf{F} = (x + y^2)\,\mathbf{i} - 2x\,\mathbf{j} + 2yz\,\mathbf{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

3. Evaluate $\iint_S \mathbf{A} \bullet \mathbf{n} \, dS$, where $\mathbf{A} = xy\,\mathbf{i} - x^2\,\mathbf{j} + (x + z)\,\mathbf{k}$, S is the portion of the plane $2x + 2y + z = 6$ included in the first octant and **n** is a unit normal to S.

4. If $\mathbf{F} = 2y\mathbf{i} - 3\,\mathbf{j} + x^2\,\mathbf{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, then evaluate $\iint_S \mathbf{F} \bullet \mathbf{n} \, dS$.

5. Evaluate $\iint_S \mathbf{F} \bullet \mathbf{n} \, dS$, where $\mathbf{F} = y\mathbf{i} + 2x\mathbf{j} - z\,\mathbf{k}$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.

6. Evaluate $\iiint_V \mathbf{F} \, dV$ where $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and V is the region bounded by the surfaces $x = 0$, $x = 2$, $y = 0$, $y = 6$, $z = 4$ and $z = x^2$.

Answers 2

1. (i) 24 (ii) 128

2. 81 3. $\dfrac{27}{4}$ 4. 132

5. 108 6. $24\,\mathbf{i} + 96\,\mathbf{j} + \dfrac{384}{5}\,\mathbf{k}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If $\mathbf{F} = x^2\,\mathbf{i} + y^3\,\mathbf{j}$ and curve C is the arc of the parabola $y = x^2$ in the xy-plane from (0, 0) to (1, 1), then $\int_C \mathbf{F} \bullet d\,\mathbf{r}$ is

(a) 7 / 12 (b) 5 / 12

(c) 11/12 (d) none of these

2. The work done in moving a particle in a force field

$$\mathbf{F} = 3x^2\ \mathbf{i} + (2xz - y)\ \mathbf{j} + 3\mathbf{k}$$

along the line joining $(0, 0, 0)$ to $(2, 1, 3)$ is

(a) 12 (b) 16

(c) 0 (d) 20

3. For a closed surface S, the value of $\iint_S \mathbf{r} \bullet \mathbf{n}\ dS$ is

(a) V (b) $2V$

(c) $3V$ (d) 0

where V is the volume enclosed by S. **(Kumaun 2013, 15)**

4. The value of $\int (x\,dy - y\,dx)$ around the circle $x^2 + y^2 = 1$ is

(a) 4π (b) 2π

(c) 0 (d) none of these

(Kanpur 2006, 12)

5. If $\mathbf{F} = 3xy\ \mathbf{i} - y^2\ \mathbf{j}$, the value of $\int_C \mathbf{F} \bullet d\mathbf{r}$ is ......, where C is the curve in the xy-plane, $y = 2x^2$, from $(0, 0)$ to $(1, 2)$

(a) 1 (b) -1

(c) 0 (d) $-7/6$ **(Kanpur 2009)**

6. The value of $\int_C \mathbf{t} \bullet d\mathbf{r}$ is ......, where t is the unit tangent vector and C is the unit circle, in xy-plane, with centre at the origin

(a) 2π (b) -2π

(c) π (d) $-\pi$

7. The value of $\int_C \mathbf{F} \bullet d\mathbf{r}$ is ......, where $\mathbf{F} = \mathbf{i} \cos y - \mathbf{j}\, x \sin y$ and C is the curve $y = \sqrt{(1 - x^2)}$ in the xy-plane from $(1, 0)$ to $(0, 1)$

(a) 1 (b) -1

(c) 0 (d) none of these

8. The value of $\int_C \mathbf{F} \bullet d\mathbf{r}$ is ......, where $\mathbf{F} = x^2 y^2 \mathbf{i} + y\,\mathbf{j}$ and C is $y^2 = 4x$ in the xy-plane from $(0, 0)$ to $(4, 4)$

(a) 268 (b) 265

(c) 264 (d) none of these

9. If $\mathbf{F} = (3x^2 + 6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, then the value of $\int_C \mathbf{F} \bullet d\mathbf{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve $x = t, y = t^2, z = t^3$ is

(a) 7 (b) 6

(c) 5 (d) none of these

10. The value of $\int_C [(x^2 + y^2)\,\mathbf{i} + (x^2 - y^2)\,\mathbf{j}] \bullet d\mathbf{r}$ where C is the curve $x^7 = y$ joining $(0, 0)$ to $(1, 1)$ is

(a) $\dfrac{38}{45}$ (b) $\dfrac{30}{45}$

(c) $\dfrac{35}{45}$ (d) none of these

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. Any integral which is to be evaluated along a curve is called a …… .
2. Any integral which is to be evaluated over a surface is called a …… .
3. If $\mathbf{r}$ denotes the position vector of a point on a curve C, then $\dfrac{d\mathbf{r}}{ds}$ is a unit vector along the …… to C at the point $\mathbf{r}$ in the direction of s increasing.
4. $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$ is called the …… of $\mathbf{F}$ over S.
5. If $\mathbf{t}$ is the unit tangent vector and C is the unit circle in $x\,y$-plane, with centre at the origin, then $\int_C \mathbf{t} \bullet d\mathbf{r} =$ …… .
6. The value of $\int (x\,dy - y\,dx)$ around the circle $x^2 + y^2 = 1$ is …… .
7. If $\mathbf{F} = (3x^2 + 6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$ and C is a straight line joining $(0, 0, 0)$ to $(1, 1, 1)$, then $\int_C \mathbf{F} \bullet d\,\mathbf{r} =$ …… .

True or False

Write 'T' for true and 'F' for false statement.

1. If C is a simple closed curve, then $\oint_C \mathbf{F} \bullet d\,\mathbf{r}$ is called the circulation of $\mathbf{F}$ about C.
2. If $\mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$, a, b, c are constants, then $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \dfrac{2}{3}\pi\,(a + b + c)$, where S is the surface of a unit sphere.

Answers

Multiple Choice Questions

1. (a)
2. (b)
3. (c)
4. (b)
5. (d)
6. (a)
7. (a)
8. (c)
9. (c)
10. (a)

Fill in the Blank(s)

1. line integral
2. surface integral
3. tangent
4. flux
5. 2π
6. 2π
7. $13/3$

True or False

1. *T*
2. *F*

□

Chapter 12

Green's, Gauss's and Stoke's Theorems

1 Green's Theorem in the Plane

Let R be a closed bounded region in the x-y plane whose boundary C consists of finitely many smooth curves. Let M and N be continuous functions of x and y having continuous partial derivatives $\dfrac{\partial M}{\partial y}$ *and* $\dfrac{\partial N}{\partial x}$ *in R. Then*

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = \oint_C (M\, dx + N\, dy),$$

the line integral being taken along the entire boundary C of R such that R is on the left as one advances in the direction of integration.

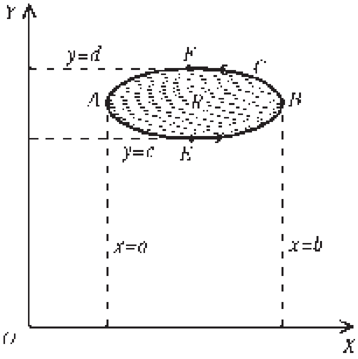
(Meerut 2004, 05, 07, 10; Avadh 10; Purvanchal 08; Rohilkhand 09B, 12; Kashi 13)

Proof: We shall first prove the theorem for a special region R bounded by a closed curve C and having the property that any straight line parallel to any one of the coordinate axes and intersecting R has only one segment (or a single point) in common with R. This means that R can be represented in both of the forms

$$a \le x \le b, \quad f(x) \le y \le g(x)$$

and $$c \le y \le d, \quad p(y) \le x \le q(y).$$

In the adjoining figure, the equations of the curves AEB and BFA are $y = f(x)$ and $y = g(x)$ respectively. Similarly the equations of the curves FAE and EBF are $x = p(y)$ and $x = q(y)$ respectively.



We have

$$\iint_R \frac{\partial M}{\partial y}\,dx\,dy$$

$$= \int_{x=a}^{b} \left[\int_{y=f(x)}^{g(x)} \frac{\partial M}{\partial y}\,dy\right] dx$$

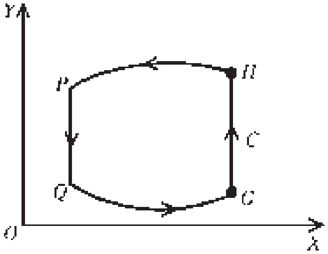
$$= \int_{x=a}^{b} \Big[M(x, y) \Big]_{y=f(x)}^{y=g(x)} dx$$

$$= \int_{x=a}^{b} \big[M[x, g(x)] - M[x, f(x)]\big]\,dx$$

$$= -\int_{a}^{b} M[x, f(x)]\,dx - \int_{b}^{a} M[x, g(x)]\,dx$$

$= -\oint_C M(x, y)\,dx$, since $y = f(x)$ represents the curve AEB and $y = g(x)$ represents the curve BFA.

If portions of C are segments parallel to y-axis such as GH and PQ in the adjoining figure, then the above result is not affected. The line integral $\int M\,dx$ over GH is zero because on GH, we have x = constant implies $dx = 0$. Similarly the line integral over PQ is zero. The equations of QG and HP are $y = f(x)$ and $y = g(x)$ respectively. Hence we have



$$-\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \oint_C M(x, y)\,dx. \qquad \ldots(1)$$

Similarly, $$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_{y=c}^{d} \left[\int_{x=p(y)}^{q(y)} \frac{\partial N}{\partial x}\,dx\right] dy$$

$$= \int_{y=c}^{d} \Big[N(x, y) \Big]_{x=p(y)}^{x=q(y)} dy$$

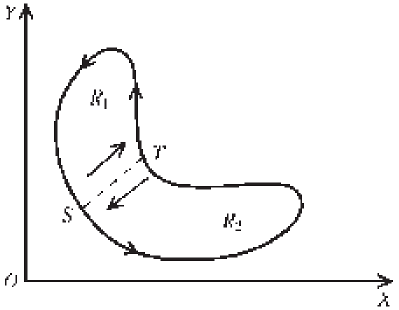
$$= \int_{y=c}^{d} \big[N[q(y), y] - N[p(y), y]\big]\,dy$$

$$= \int_{c}^{d} N[q(y), y]\,dy + \int_{d}^{c} N[p(y), y]\,dy$$

$$= \oint_C N(x, y)\,dy. \qquad \ldots(2)$$

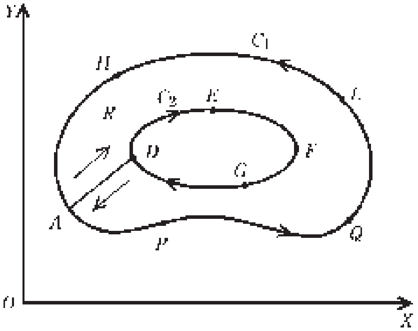
From (1) and (2), we get on adding $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C (M\, dx + N\, dy)$.

The proof of the theorem can now be extended to a region R which can be subdivided into finitely many special regions of the above type by drawing lines (TS in the adjoining figure). In this case we apply the theorem to each subregion (R_1 and R_2 in the figure) and then add the results. The sum of the left hand members will be equal to the integral over R. The sum of the right hand members will be equal to the integral over C plus the line integrals over the curves introduced for subdividing R. Each of the latter integrals comes twice, taken once in each direction (as ST and TS in the figure). Therefore these two integrals cancel each other and thus the sum of the right hand members will be equal to the line integral over C.



Note: **Extension of Green's theorem in plane to multiply connected regions.**

Green's theorem in the plane is also valid for a multiply-connected region R such as shown in the adjoining figure. Here the boundary C of R consists of two parts; the exterior boundary C_1 is traversed in the anticlockwise sense so that R is on the left, while the interior boundary C_2 is traversed in the clockwise sense so that R is on the left.



In order to establish the theorem, we construct a line such as AD (called a *cross cut*) connecting the exterior and interior boundaries. The region bounded by $ADEFGDAPQLHA$ is simply-connected and so Green's theorem is valid for it. Therefore

$$\oint_{\substack{C \\ ADEFGDAPQLHA}} M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$$

The integral on the left hand side leaving out the integrand is equal to

$$\int_{AD} + \int_{C_2} + \int_{DA} + \int_{C_1} = \int_{C_2} + \int_{C_1} \text{, since } \int_{AD} = -\int_{DA}$$

$$= \oint_C (M\, dx + N\, dy).$$

Hence the theorem.

2 Green's Theorem in the Plane in Vector Notation

We have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ so that $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$.

Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$.

Then $M\,dx + N\,dy = (M\,\mathbf{i} + N\,\mathbf{j}) \bullet (dx\,\mathbf{i} + dy\,\mathbf{j}) = \mathbf{F} \bullet d\mathbf{r}$.

Also $\text{curl}\mathbf{F} = \nabla \times \mathbf{F}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = -\frac{\partial N}{\partial z}\,\mathbf{i} + \frac{\partial M}{\partial z}\,\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

$$\therefore \qquad (\nabla \times \mathbf{F}) \bullet \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

Hence Green's theorem in plane can be written as

$$\iint_R (\nabla \times \mathbf{F}) \bullet \mathbf{k}\; dR = \oint_C \mathbf{F} \bullet d\mathbf{r}$$

where $dR = dx\,dy$ and $\mathbf{k}$ is unit vector perpendicular to the xy-plane.

If s denotes the arc length of C and $\mathbf{t}$ is the unit tangent vector to C, then $d\mathbf{r} = \dfrac{d\mathbf{r}}{ds}\,ds = \mathbf{t}\,ds$. Therefore the above result can also be written as

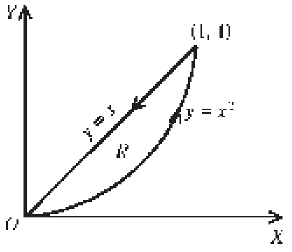
$$\iint_R (\nabla \times \mathbf{F}) \bullet \mathbf{k}\; dR = \oint_C \mathbf{F} \bullet \mathbf{t}\; ds.$$

Illustrative Examples

Example 1: *Verify Green's theorem in the plane for* $\oint_C (xy + y^2)\,dx + x^2\,dy$ *, where C is the closed curve of the region bounded by* $y = x$ *and* $y = x^2$*.* **(Rohilkhand 2011)**

Solution: By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy).$$



Here $M = xy + y^2$, $N = x^2$.

The curves $y = x$ and $y = x^2$ intersect at $(0, 0)$ and $(1, 1)$. The positive direction in traversing C is as shown in the figure.

We have
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$
$$= \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\,dy$$
$$= \iint_R (2x - x - 2y)\,dx\,dy = \iint_R (x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x-2y)\,dy\,dx = \int_{x=0}^{1} \left[xy - y^2 \right]_{y=x^2}^{x} dx$$

$$= \int_0^1 [x^2 - x^2 - x^3 + x^4]\,dx = \int_0^1 (x^4 - x^3)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}\cdot$$

Now let us evaluate the line integral along C. Along $y = x^2$, $dy = 2x\,dx$. Therefore along $y = x^2$, the line integral equals

$$\int_0^1 [\{(x)(x^2) + x^4\}\,dx + x^2 (2x)\,dx] = \int_0^1 (3x^3 + x^4)\,dx = \frac{19}{20}\cdot$$

Along $y = x$, $dy = dx$. Therefore along $y = x$, the line integral equals

$$\int_1^0 [\{(x)(x) + x^2\}\,dx + x^2\,dx] = \int_1^0 3x^4\,dx = -1.$$

Therefore the required line integral $= \dfrac{19}{20} - 1 = -\dfrac{1}{20}\cdot$ Hence the theorem is verified.

Example 2: *Evaluate by Green's theorem* $\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$, *where C is the rectangle with vertices* $(0, 0)$, $(\pi, 0)$, $(\pi, 1)$, $(0, 1)$.

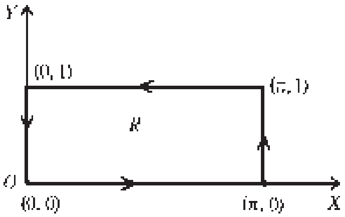
(Meerut 2002, 05B, 06, 13B; Rohilkhand 14)

Solution: By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy).$$

Here $M = x^2 - \cosh y$, $N = y + \sin x$.

$$\therefore \qquad \frac{\partial N}{\partial x} = \cos x, \quad \frac{\partial M}{\partial y} = -\sinh y.$$



Hence the given line integral is equal to

$$\iint_R (\cos x + \sinh y)\,dx\,dy$$

$$= \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)\,dy\,dx$$

$$= \int_{x=0}^{\pi} \left[y \cos x + \cosh y \right]_{y=0}^{1} dx$$

$$= \int_{x=0}^{\pi} [\cos x + \cosh 1 - 1]\,dx = \left[\sin x + x \cosh 1 - x \right]_0^{\pi} = (\cosh 1 - 1).$$

Example 3: *Show that the area bounded by a simple closed curve C is given by* $\dfrac{1}{2} \oint_C (x\,dy - y\,dx)$. *Hence find the area of the ellipse* $x = a \cos \theta$, $y = b \sin \theta$.

Solution: By Green's theorem in plane, if R is a plane region bounded by a simple closed curve C, then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C M\, dx + N\, dy.$$

Putting $M = -y,\ N = x$, we get

$$\oint_C (x\, dy - y\, dx) = \iint_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right] dx\, dy$$

$$= 2 \iint_R dx\, dy$$

$= 2A$, where A is the area bounded by C.

Hence $\quad A = \frac{1}{2}\oint_C (x\, dy - y\, dx).$

The area of the ellipse

$$= \frac{1}{2}\oint_C (x\, dy - y\, dx) = \frac{1}{2}\int_{\theta=0}^{2\pi} \left(a\cos\theta \frac{dy}{d\theta} - b\sin\theta \frac{dx}{d\theta}\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (ab\cos^2\theta + ab\sin^2\theta)\, d\theta = \frac{1}{2}ab\int_0^{2\pi} d\theta = \pi ab.$$

Comprehensive Exercise 1

1. Verify Green's theorem in the plane for
$\int_C [(2xy - x^2)\, dx + (x^2 + y^2)\, dy]$,
where C is the boundary of the region enclosed by $y = x^2$ and $y^2 = x$ described in the positive sense.
2. Verify Green's theorem in the plane for
$\oint_C [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$,
where C is the boundary of the region defined by $y = \sqrt{x},\ y = x^2$.
3. Apply Green's theorem in the plane to evaluate
$\int_C \{(y - \sin x)\, dx + \cos x\, dy\}$,
where C is the triangle enclosed by the lines $y = 0,\ x = 2\pi,\ \pi y = 2x$.
(Avadh 2010)
4. Evaluate by Green's theorem in plane $\int_C (e^{-x}\sin y\, dx + e^{-x}\cos y\, dy)$,
where C is the rectangle with vertices $(0, 0), (\pi, 0), \left(\pi, \frac{1}{2}\pi\right), \left(0, \frac{1}{2}\pi\right)$.
5. Evaluate by Green's theorem $\oint_C (\cos x \sin y - xy)\, dx + \sin x \cos y\, dy$,
where C is the circle $x^2 + y^2 = 1$.
(Kumaun 2012)
6. If $\mathbf{F} = (x^2 - y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, find the value of $\int \mathbf{F} \bullet d\mathbf{r}$ around the rectangular boundary $x = 0,\ x = a,\ y = 0,\ y = b$.

7. Verify Green's theorem in the plane for

$$\int_C (x^2 - xy^3)\, dx + (y^2 - 2xy)\, dy,$$

where C is the square with vertices $(0, 0),\ (2, 0),\ (2, 2),\ (0, 2)$.

(Meerut 2001)

8. Apply Green's theorem in the plane to evaluate $\int_C [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy]$, where C is the boundary of the surface enclosed by the x-axis and the semi-circle $y = (1 - x^2)^{1/2}$.

9. If C is the simple closed curve in the xy-plane not enclosing the origin, show that $\int_C \mathbf{F} \bullet d\mathbf{r} = 0$, where $\mathbf{F} = \dfrac{-\mathbf{i}\, y + \mathbf{j}\, x}{x^2 + y^2}$.

Answers 1

3. $-\dfrac{\pi}{4} - \dfrac{2}{\pi}$

4. $2\,(e^{-\pi} - 1)$

5. 0

6. $2ab^2$

3 The Divergence Theorem of Gauss

Suppose V is the volume bounded by a closed piecewise smooth surface S. Suppose $\mathbf{F}(x, y, z)$ *is a vector function of position which is continuous and has continuous first partial derivatives in V. Then* $\iiint_V \nabla \bullet \mathbf{F}\, dV = \iint_S \mathbf{F} \bullet \mathbf{n}\, dS,$

where $\mathbf{n}$ *is the outward drawn unit normal vector to S.*

(Meerut 2000, 01, 06, 10B, 12, 12B;
Bundelkhand 09, 11; Avadh 14; Kashi 14)

Since $\mathbf{F} \bullet \mathbf{n}$ is the normal component of vector $\mathbf{F}$, therefore divergence theorem may also be stated as follows :

The surface integral of the normal component of a vector **F** *taken over a closed surface is equal to the integral of the divergence of* **F** *taken over the volume enclosed by the surface.*

Cartesian equivalent of Divergence Theorem:

Let $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$.

Then $\nabla \bullet \mathbf{F} = \text{div } \mathbf{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$.

If α, β, γ are the angles which outward drawn unit normal $\mathbf{n}$ makes with positive directions of x, y, z-axes, then $\cos\alpha, \cos\beta, \cos\gamma$ are direction cosines of $\mathbf{n}$ and we have $\mathbf{n} = \cos\alpha\, \mathbf{i} + \cos\beta\, \mathbf{j} + \cos\gamma\, \mathbf{k}$.

$$\therefore \quad \mathbf{F} \bullet \mathbf{n} = (F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}) \bullet (\cos\alpha\, \mathbf{i} + \cos\beta\, \mathbf{j} + \cos\gamma\, \mathbf{k})$$
$$= F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma.$$

Therefore the divergence theorem can be written as

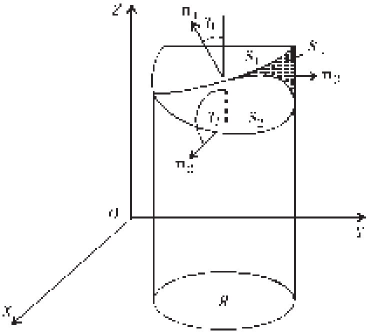
$$\iiint_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) dx\ dy\ dz$$

$$= \iint_S \ (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\ dS$$

$$= \iint \ F_1\ dy\ dz + F_2\ dz\ dx + F_3\ dx\ dy).$$

The significance of divergence theorem lies in the fact that a surface integral may be expressed as a volume integral and vice versa.

Proof of the divergence theorem:



We shall first prove the theorem for a special region V which is bounded by a piecewise smooth closed surface S and has the property that any straight line parallel to any one of the coordinate axes and intersecting V has only one segment (or a single point) in common with V. If R is the orthogonal projection of S on the xy-plane, then V can be represented in the form $f\ (x, y) \le z \le g\ (x, y)$ where (x, y) varies in R.

Obviously $z = g\ (x, y)$ represents the upper portion S_1 of S, $z = f\ (x, y)$ represents the lower portion S_2 of S and there may be a remaining vertical portion S_3 of S.

We have

$$\iiint_V \ \frac{\partial F_3}{\partial z}\ dV = \iiint_V \ \frac{\partial F_3}{\partial z}\ dx\ dy\ dz$$

$$= \iint_R \left[\int_{z=f\ (x,\ y)}^{g\ (x,\ y)} \frac{\partial F_3}{\partial z}\ dz\right] dx\ dy$$

$$= \iint_R \ \left[F_3\ (x, y, z)\right]_{z=f\ (x,\ y)}^{g\ (x,\ y)} dx\ dy$$

$$= \iint_R \ \left[F_3\ [x, y, g\ (x, y)] - F_3\ [x, y, f\ (x, y)]\right] dx\ dy$$

$$= \iint_R \ F_3\ [\ x,\ \ y,\ \ g\ (x, y)]\ dx\ dy$$

$$- \iint_R \ F_3\ [\ x,\ \ y,\ \ f\ (x, y)]\ dx\ dy \qquad \text{...(1)}$$

Now for the vertical portion S_3 of S, the normal $\mathbf{n}_3$ to S_3 makes a right angle γ with $\mathbf{k}$. Therefore $\iint_{S_3} \ F_3\ \mathbf{k} \bullet \mathbf{n}_3\ dS_3 = 0$, since $\mathbf{k} \bullet \mathbf{n}_3 = 0$.

For the upper portion S_1 of S, the normal $\mathbf{n}_1$ to S_1 makes an acute angle γ_1 with $\mathbf{k}$.

Therefore $\quad \mathbf{k} \bullet \mathbf{n}_1\ dS_1 = \cos\gamma_1\ dS_1 = dx\ dy.$

Hence $\quad \iint_{S_1} \ F_3\ \mathbf{k} \bullet \mathbf{n}_1\ dS_1 = \iint_R \ F_3\ [\ x,\ \ y,\ \ g\ (x, y)]\ dx\ dy.$

For the lower portion S_2 of S, the normal $\mathbf{n}_2$ to S_2 makes an obtuse angle γ_2 with $\mathbf{k}$.

Therefore $\quad \mathbf{k} \bullet \mathbf{n}_2 \, dS_2 = \cos \gamma_2 \, dS_2 = - \, dx \, dy.$

Hence $\quad \iint_{S_1} F_3 \, \mathbf{k} \bullet \mathbf{n}_2 \, dS_2 = - \iint_R F_3 \, [x \;\; y, \; f\,(x, y)] \, dx \, dy.$

$$\therefore \quad \iint_S F_3 \, \mathbf{k} \bullet \mathbf{n}_3 \, dS_3 + \iint_{S_1} F_3 \, \mathbf{k} \bullet \mathbf{n}_1 \, dS_1 + \iint_{S_2} F_3 \, \mathbf{k} \bullet \mathbf{n}_2 \, dS_2$$

$$= 0 + \iint_R F_3 \, [\, x, \; y, \; g \; (x, \; y)] \, dx \, dy$$

$$- \iint_R F_3 \, [\, x, \; y, \; f\,(x, y)] \, dx \, dy$$

or with the help of (1), we get

$$\iint_S F_3 \, \mathbf{k} \bullet \mathbf{n} \, dS = \iiint_V \frac{\partial F_3}{\partial z} \, dV. \qquad \text{...(2)}$$

Similarly, by projecting S on the other co-ordinate planes, we get

$$\iint_S F_2 \, \mathbf{j} \bullet \mathbf{n} \, dS = \iiint_V \frac{\partial F_2}{\partial y} \, dV \qquad \text{...(3)}$$

and

$$\iint_S F_1 \, \mathbf{i} \bullet \mathbf{n} \, dS = \iiint_V \frac{\partial F_1}{\partial x} \, dV \qquad \text{...(4)}$$

Adding (2), (3) and (4), we get

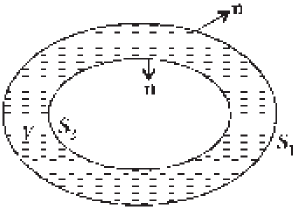
$$\iint_S (F_1 \, \mathbf{i} + F_2 \, \mathbf{j} + F_3 \, \mathbf{k}) \bullet \mathbf{n} \, dS = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dV$$

or

$$\iiint_V \nabla \bullet \mathbf{F} \, dV = \iint_S \mathbf{F} \bullet \mathbf{n} \, dS.$$

The proof of the theorem can now be extended to a region V which can be subdivided into finitely many special regions of the above type by drawing auxiliary surfaces. In this case we apply the theorem to each sub-region and then add the results. The sum of the volume integrals over parts of V will be equal to the volume integral over V. The surface integrals over auxiliary surfaces cancel in pairs, while the sum of the remaining surface integrals is equal to the surface integral over the whole boundary S of V.

Note: The divergence theorem is applicable for a region V if it is bounded by two closed surfaces S_1 and S_2 one of which lies within the other. Here outward drawn normals will have the directions as shown in the figure.



4 Some Deductions from Divergence Theorem

1. Green's theorem: *Let* ϕ *and* ψ *be scalar point functions which together with their derivatives in any direction are uniform and continuous within the region V bounded by a closed surface S, then*

$$\iiint_V (\phi\nabla^2 \psi - \psi\nabla^2 \phi)\, dV = \iint_S (\phi\nabla \psi - \psi\nabla \phi) \bullet \mathbf{n}\, dS.$$

Proof: By divergence theorem, we have $\iiint_V \nabla \bullet \mathbf{F}\, dV = \iint_S \mathbf{F} \bullet \mathbf{n}\, dS.$

Putting $\mathbf{F} = \phi\nabla\psi$, we get $\nabla \bullet \mathbf{F} = \nabla \bullet (\phi\nabla\psi)$

$$= \phi(\nabla \bullet \nabla\psi) + (\nabla\phi) \bullet (\nabla\psi) = \phi\nabla^2 \psi + (\nabla\phi) \bullet (\nabla\psi).$$

Also $\mathbf{F} \bullet \mathbf{n} = (\phi\nabla\psi) \bullet \mathbf{n}.$

∴ divergence theorem gives

$$\iiint_V [\phi\nabla^2 \psi + (\nabla\phi) \bullet (\nabla\psi)]\, dV = \iint_S (\phi\nabla\psi) \bullet \mathbf{n}\, dS \qquad ...(1)$$

This is called *Green's first identity* or *theorem.*

Interchanging ϕ and ψ in (1), we get

$$\iiint_V [\psi\nabla^2 \phi + (\nabla\psi) \bullet (\nabla\phi)]\, dV = \iint_S [\psi\nabla\phi] \bullet \mathbf{n}\, dS \qquad ...(2)$$

Subtracting (2) from (1), we get

$$\iiint_V (\phi\nabla^2 \psi - \psi\nabla^2 \phi)\, dV = \iint_S (\phi\nabla \psi - \psi\nabla\phi) \bullet \mathbf{n}\, dS \qquad ...(3)$$

This is called *Green's second identity* or *Green's theorem in symmetrical form.*

Since $\nabla\psi = \dfrac{\partial\psi}{\partial n}\mathbf{n}$ and $\nabla\phi = \dfrac{\partial\phi}{\partial n}\mathbf{n}$, therefore

$$(\phi\nabla\psi - \psi\nabla\phi) \bullet \mathbf{n} = \left(\phi \frac{\partial\psi}{\partial n}\mathbf{n} - \psi \frac{\partial\phi}{\partial n}\mathbf{n}\right) \bullet \mathbf{n}$$

$$= \phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n}.$$

Hence (3) can also be written as

$$\iiint_V (\phi\nabla^2 \psi - \psi\nabla^2 \phi)\, dV = \iint_S \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n}\right) dS.$$

Note: **Harmonic function:** *If a scalar point function* ϕ *satisfies Laplace's equation* $\nabla^2 \phi = 0$, *then* ϕ *is called harmonic function.* If ϕ and ψ are both harmonic functions, then $\nabla^2 \phi = 0,\ \nabla^2 \psi = 0.$

Hence from Green's second identity, we get $\iint_S \left(\phi \dfrac{\partial\psi}{\partial n} - \psi \dfrac{\partial\phi}{\partial n}\right) dS = 0.$

2. *Prove that* $\iiint_V \nabla\phi\, dV = \iint_S \phi\mathbf{n}\, dS.$

Proof: By divergence theorem, we have $\iiint_V \nabla \bullet \mathbf{F}\, dV = \iint_S \mathbf{F} \bullet \mathbf{n}\, dS.$

Taking $\mathbf{F} = \phi \mathbf{C}$ where $\mathbf{C}$ is an arbitrary constant non-zero vector, we get

$$\iiint_V \nabla \bullet (\phi \mathbf{C})\, dV = \iint_S (\phi \mathbf{C}) \bullet \mathbf{n}\, dS. \qquad \ldots(1)$$

Now $\nabla \bullet (\phi \mathbf{C}) = (\nabla \phi) \bullet \mathbf{C} + \phi (\nabla \bullet \mathbf{C}) = (\nabla \phi) \bullet \mathbf{C}$, since $\nabla \bullet \mathbf{C} = 0$.

Also $(\phi \mathbf{C}) \bullet \mathbf{n} = \mathbf{C} \bullet (\phi\, \mathbf{n})$.

$\therefore$ (1) becomes

$$\iiint_V \mathbf{C} \bullet (\nabla \phi)\, dV = \iint_S \mathbf{C} \bullet (\phi \mathbf{n})\, dS$$

or
$$C \bullet \iiint_V \nabla \phi\, dV = \mathbf{C} \bullet \int_S (\phi \mathbf{n})\, dS$$

or
$$\mathbf{C} \bullet \left[\iiint_V \nabla \phi\, dV - \iint_S \phi \mathbf{n}\, dS \right] = 0.$$

Since $\mathbf{C}$ is an arbitrary vector, therefore we must have

$$\iiint_V \nabla \phi\, dV = \iint_S \phi \mathbf{n}\, dS.$$

3. *Prove that* $\iiint_V \nabla \times \mathbf{B}\, dV = \iint_S \mathbf{n} \times \mathbf{B}\, dS.$

Proof: In divergence theorem taking $\mathbf{F} = \mathbf{B} \times \mathbf{C}$**,** where $\mathbf{C}$ is an arbitrary constant vector, we get

$$\iiint_V \nabla \bullet (\mathbf{B} \times \mathbf{C})\, dV = \iint_S (\mathbf{B} \times \mathbf{C}) \bullet \mathbf{n}\, dS. \qquad \ldots(1)$$

Now $\nabla \bullet (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \bullet \text{curl } \mathbf{B} - \mathbf{B} \bullet \text{curl } \mathbf{C} = \mathbf{C} \bullet \text{curl } \mathbf{B}$, since curl $\mathbf{C} = \mathbf{0}$.

Also $(\mathbf{B} \times \mathbf{C}) \bullet \mathbf{n} = [\mathbf{B}, \mathbf{C}, \mathbf{n}] = [\mathbf{C}, \mathbf{n}, \mathbf{B}] = \mathbf{C} \bullet (\mathbf{n} \times \mathbf{B})$.

$\therefore$ (1) becomes

$$\iiint_V (\mathbf{C} \bullet \text{curl } \mathbf{B})\, dV = \iint_S \mathbf{C} \bullet (\mathbf{n} \times \mathbf{B})\, dS$$

or
$$\mathbf{C} \bullet \iiint_V (\nabla \times \mathbf{B})\, dV = \mathbf{C} \bullet \iint_S (\mathbf{n} \times \mathbf{B})\, dS$$

or
$$\mathbf{C} \bullet \left[\iiint_V (\nabla \times \mathbf{B})\, dV - \iint_S (\mathbf{n} \times \mathbf{B})\, dS \right] = 0.$$

Since $\mathbf{C}$ is an arbitrary vector therefore we can take $\mathbf{C}$ as a non-zero vector which is not perpendicular to the vector

$$\iiint_V (\nabla \times \mathbf{B})\, dV - \iint_S (\mathbf{n} \times \mathbf{B})\, dS.$$

Hence we must have

$$\iiint_V (\nabla \times \mathbf{B})\, dV - \iint_S (\mathbf{n} \times \mathbf{B})\, dS = \mathbf{0}$$

or
$$\iiint_V (\nabla \times \mathbf{B})\, dV = \iint_S (\mathbf{n} \times \mathbf{B})\, dS.$$

Illustrative Examples

Example 4: *(i) For any closed surface S, prove that* $\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\, dS = 0.$

(Meerut 2009B; Purvanchal 14; Avadh 14)

(ii) Evaluate $\iint_S \mathbf{r} \bullet \mathbf{n}\, dS$, *where S is a closed surface.*

(Agra 2006; Kumaun 07; Bundelkhand 09; Purvanchal 14)

(iii) If $\mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$, *a, b, c are constants show that*

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, d\mathbf{S} = \frac{4}{3}\pi\,(a+b+c),$$

where S is the surface of a unit sphere.

(Bundelkhand 2001, 07, 08; Rohilkhand 07; Kashi 13)

Solution: (i) By divergence theorem, we have

$$\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\, dS = \iiint_V (\text{div curl } \mathbf{F})\, dV,$$

where V is the volume enclosed by S

$$= 0, \text{ since div curl } \mathbf{F} = 0.$$

(ii) By the divergence theorem, we have

$$\iint_S \mathbf{r} \bullet \mathbf{n}\, dS = \iiint_V \nabla \bullet \mathbf{r}\, dV = \iiint_V 3\, dV,$$

since $\nabla \bullet \mathbf{r} = \text{div } \mathbf{r} = 3$

$$= 3V, \text{ where } V \text{ is the volume enclosed by } S.$$

(iii) By the divergence theorem, we have

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \iiint_V (\nabla \bullet \mathbf{F})\, dV,$$

where V is the volume enclosed by S

$$= \iiint_V [\nabla \bullet (a\,x\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k})]\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(a\,x) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\right] dV$$

$$= \iiint_V (a+b+c)\, dV$$

$$= (a+b+c)\,V = (a+b+c)\,\frac{4}{3}\pi,$$

since the volume V enclosed by a sphere of unit radius is equal to $\frac{4}{3}\pi(1)^3$ *i.e.,* $\frac{4}{3}\pi$.

Example 5(i): *Show that* $\iint_S \mathbf{n}\, dS = \mathbf{0}$ *for any closed surface S.* **(Purvanchal 14)**

(ii) Prove that $\iint_S \mathbf{r} \times \mathbf{n}\, dS = \mathbf{0}$ *for any closed surface S.* **(Agra 2007)**

(iii) Prove that $\iint_S \mathbf{n} \times (\mathbf{a} \times \mathbf{r})\, dS = 2V\mathbf{a}$, *where* **a** *is a constant vector and V is the volume enclosed by the closed surface S.* **(Avadh 2011, 12)**

Solution: (i) Let **C** be any arbitrary constant vector.

Then $$\mathbf{C} \bullet \iint_S \mathbf{n}\, dS = \iint_S \mathbf{C} \bullet \mathbf{n}\, dS$$

$$= \iiint_V (\nabla \bullet \mathbf{C})\, dV, \text{ by divergence theorem}$$

$$= 0, \quad \text{since div } \mathbf{C} = 0.$$

Thus $\quad \mathbf{C} \bullet \iint_S \mathbf{n}\, dS = 0$, where $\mathbf{C}$ is an arbitrary vector.

Therefore we must have $\iint_S \mathbf{n}\, dS = \mathbf{0}$.

(ii) Let $\mathbf{C}$ be any arbitrary constant vector. Then

$$\mathbf{C} \bullet \iint_S \mathbf{r} \times \mathbf{n}\, dS = \iint_S \mathbf{C} \bullet [(\mathbf{r} \times \mathbf{n})]\, dS = \iint_S (\mathbf{C} \times \mathbf{r}) \bullet \mathbf{n}\, dS$$

$$= \iiint_V [\nabla \bullet (\mathbf{C} \times \mathbf{r})]\, dV, \text{ by divergence theorem}$$

$$= \iiint_V [\mathbf{r} \bullet \text{curl}\, \mathbf{C} - \mathbf{C} \bullet \text{curl}\, \mathbf{r}]\, dV = 0,$$

since curl $\mathbf{C} = \mathbf{0}$ and curl $\mathbf{r} = \mathbf{0}$.

Thus $\quad \mathbf{C} \bullet \iint_S \mathbf{r} \times \mathbf{n}\, dS = 0$, where $\mathbf{C}$ is an arbitrary vector.

Therefore, we must have $\iint_S \mathbf{r} \times \mathbf{n}\, dS = \mathbf{0}$.

(iii) We know that

$$\iiint_V \nabla \times \mathbf{B}\, dV = \iint_S \mathbf{n} \times \mathbf{B}\, dS. \qquad \text{[See article 4, part 3]}$$

Putting $\mathbf{B} = \mathbf{a} \times \mathbf{r}$, we get

$$\iint_S \mathbf{n} \times (\mathbf{a} \times \mathbf{r})\, dS = \iiint_V \nabla \times (\mathbf{a} \times \mathbf{r})\, dV$$

$$= \iiint_V \text{curl}\, (\mathbf{a} \times \mathbf{r})\, dV$$

$$= \iiint_V 2\mathbf{a}\, dV, \text{ since curl } (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$$

$$= 2\mathbf{a} \iiint_V dV = 2V\mathbf{a}.$$

Example 6: *Using the divergence theorem, show that the volume V of a region T bounded by a surface S is*

$$V = \iint_S x\, dy\, dz = \iint_S y\, dz\, dx = \iint_S z\, dx\, dy$$

$$= \frac{1}{3} \iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy).$$

(Meerut 2010)

Solution: By divergence theorem, we have

$$\iint_S x\, dy\, dz = \iiint_V \left(\frac{\partial}{\partial x} (x) \right) dV = \iiint_V dV = V$$

$$\iint_S y\, dz\, dx = \iiint_V \left(\frac{\partial}{\partial y} (y) \right) dV = \iiint_V dV = V$$

$$\iint_S z\, dx\, dy = \iiint_V \left(\frac{\partial}{\partial z} (z) \right) dV = \iiint_V dV = V.$$

Adding these results, we get

$$3V = \iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$$

or

$$V = \frac{1}{3} \iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy).$$

Example 7: *Verify divergence theorem for*

$$\mathbf{F} = (x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k}$$

taken over the rectangular parallelopiped $0 \le x \le a, 0 \le y \le b, \ 0 \le z \le c$.

(Meerut 2006B; Avadh 09; Rohilkhand 13)

Solution: We have

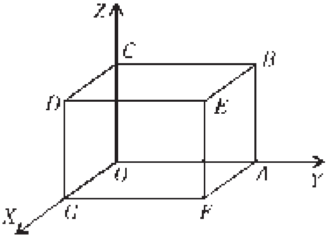
$$\begin{aligned} \text{div } \mathbf{F} &= \nabla \bullet \mathbf{F} \\ &= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) \\ &= 2x + 2y + 2z. \end{aligned}$$

$\therefore$ volume integer $= \iiint_V \nabla \bullet \mathbf{F}\, dV = \iiint_V 2(x+y+z)\, dV$

$$\begin{aligned} &= 2\int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} (x+y+z)\, dx\, dy\, dz \\ &= 2\int_{z=0}^{c} \int_{y=0}^{b} \left[\frac{x^2}{2} + yx + zx\right]_{x=0}^{a} dy\, dz \\ &= 2\int_{z=0}^{c} \int_{y=0}^{b} \left[\frac{a^2}{2} + ay + az\right] dy\, dz \\ &= 2\int_{z=0}^{c} \left[\frac{a^2}{2} y + a\frac{y^2}{2} + azy\right]_{y=0}^{b} dz \\ &= 2\int_{z=0}^{c} \left[\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right] dz \\ &= 2\left[\frac{a^2 b}{2} z + \frac{ab^2}{2} z + ab\frac{z^2}{2}\right]_0^c \\ &= [a^2 bc + ab^2 c + abc^2] = abc\,(a+b+c). \end{aligned}$$

Surface Integral: We shall now calculate $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$ over the six faces of the rectangular parallelopiped.

Over the face *DEFG*, $\mathbf{n} = \mathbf{i}$, $x = a$.



Therefore, $\iint_{DEFG} \mathbf{F} \bullet \mathbf{n}\, dS$

$$\begin{aligned} &= \int_{z=0}^{c} \int_{y=0}^{b} [(a^2 - yz)\,\mathbf{i} \\ &\qquad + (y^2 - za)\,\mathbf{j} + (z^2 - ay)\,\mathbf{k}] \bullet \mathbf{i}\, dy\, dz \\ &= \int_{z=0}^{c} \int_{y=0}^{b} (a^2 - yz)\, dy\, dz \\ &= \int_{z=0}^{c} \left[a^2 y - z\frac{y^2}{2}\right]_{y=0}^{b} dz \end{aligned}$$

$$= \int_{z=0}^{c} \left[a^2 b - \frac{zb^2}{2} \right] dz = \left[a^2 bz - \frac{z^2}{4} b^2 \right]_0^c = a^2 bc - \frac{c^2 b^2}{4} \cdot$$

Over the face $ABCO$, $\mathbf{n} = -\mathbf{i}$, $x = 0$. Therefore

$$\iint_{ABCO} \mathbf{F} \bullet \mathbf{n} \, dS = \iint [(0 - yz)\,\mathbf{i} + \ldots + \ldots] \bullet (-\mathbf{i}) \, dy \, dz$$

$$= \int_{z=0}^{c} \int_{y=0}^{b} y \, z \, dy \, dz = \int_{z=0}^{c} \left[\frac{y^2}{2} z \right]_{y=0}^{b} dz$$

$$= \int_{z=0}^{c} \frac{b^2}{2} z \, dz = \frac{b^2 c^2}{4} \cdot$$

Over the face $ABEF$, $\mathbf{n} = \mathbf{j}$, $y = b$. Therefore

$$\iint_{ABEF} \mathbf{F} \bullet \mathbf{n} \, dS = \int_{z=0}^{c} \int_{x=0}^{a} [(x^2 - bz)\,\mathbf{i} + (b^2 - zx)\,\mathbf{j} + (z^2 - bx)\,\mathbf{k}] \bullet \mathbf{j} \, dx \, dz$$

$$= \int_{z=0}^{c} \int_{x=0}^{a} (b^2 - zx) \, dx \, dz = b^2 \, ca - \frac{a^2 c^2}{4} \cdot$$

Over the face $OGDC$, $\mathbf{n} = -\mathbf{j}$, $y = 0$. Therefore

$$\iint_{OGDC} \mathbf{F} \bullet \mathbf{n} \, dS = \int_{z=0}^{c} \int_{x=0}^{a} zx \, dx \, dz = \frac{c^2 a^2}{4} \cdot$$

Over the face $BCDE$, $\mathbf{n} = \mathbf{k}$, $z = c$. Therefore

$$\iint_{BCDE} \mathbf{F} \bullet \mathbf{n} \, dS = \int_{y=0}^{b} \int_{x=0}^{a} (c^2 - xy) \, dx \, dy = c^2 ab - \frac{a^2 b^2}{4} \cdot$$

Over the face $AFGO$, $\mathbf{n} = -\mathbf{k}$, $z = 0$. Therefore

$$\iint_{AFGO} \mathbf{F} \bullet \mathbf{n} \, dS = \int_{y=0}^{b} \int_{x=0}^{a} x \, y \, dx \, dy = \frac{a^2 \, b^2}{4} \cdot$$

Adding the six surface integrals, we get

$$\iint_S \mathbf{F} \bullet \mathbf{n} \, dS = \left(a^2 bc - \frac{c^2 b^2}{4} + \frac{c^2 b^2}{4} \right) + \left(b^2 ca - \frac{a^2 c^2}{4} + \frac{a^2 c^2}{4} \right) + \left(c^2 ab - \frac{a^2 b^2}{4} + \frac{a^2 b^2}{4} \right)$$

$$= abc \, (a + b + c).$$

Hence the theorem is verified.

Example 8: *If* $\mathbf{F} = x\,\mathbf{i} - y\,\mathbf{j} + (z^2 - 1)\mathbf{k}$, *find the value of* $\iint_S \mathbf{F} \bullet \mathbf{n} \, dS$ *where* S *is the closed surface bounded by the planes* $z = 0$, $z = 1$ *and the cylinder* $x^2 + y^2 = 4$.

(Garhwal 2000; Kanpur 05; Avadh 13; Kumaun 15)

Solution: By divergence theorem, we have $\iint_S \mathbf{F} \bullet \mathbf{n} \, dS = \iiint_V \text{div } \mathbf{F} \, dV$.

Here $$\text{div } \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(z^2 - 1) = 1 - 1 + 2z = 2z.$$

$\therefore$
$$\iiint_V \text{div } \mathbf{F}\, dV = \int_{z=0}^{1} \int_{y=-2}^{2} \int_{x=-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}} 2z\, dx\, dy\, dz$$

$$= \int_{z=0}^{1} \int_{y=-2}^{2} [2zx]_{x=-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}}\, dy\, dz$$

$$= \int_{z=0}^{1} \int_{y=-2}^{2} 4z\sqrt{(4-y^2)}\, dy\, dz$$

$$= \int_{y=-2}^{2} \left[4\frac{z^2}{2}\sqrt{(4-y^2)}\right]_{z=0}^{1} dy$$

$$= 2\int_{y=-2}^{2} \sqrt{(4-y^2)}\, dy = 4\int_0^2 \sqrt{(4-y^2)}\, dy$$

$$= 4\left[\frac{y}{2}\sqrt{(4-y^2)} + 2\sin^{-1}\frac{y}{2}\right]_0^2$$

$$= 4\,[2\sin^{-1} 1] = 4\,(2)\,\frac{\pi}{2} = 4\pi.$$

Comprehensive Exercise 2

1. (i) Verify divergence theorem for $\mathbf{F} = (2x - z)\mathbf{i} + x^2 y\mathbf{j} - xz^2\mathbf{k}$ taken over the region bounded by $x = 0, x = 2, y = 0, y = 2, z = 0, z = 2$.

(ii) Verify divergence theorem for $\mathbf{F} = (2x - z)\,\mathbf{i} + x^2 y\,\mathbf{j} - xz^2\mathbf{k}$ taken over the region bounded by $x = 0,\ x = 1,\ y = 0, y = 1, z = 0, z = 1$.

(Garhwal 2001; Kumaun 14)

2. (i) If $\mathbf{F} = 4xz\,\mathbf{i} - y^2\mathbf{j} + yz\,\mathbf{k}$ and S is the surface bounded by $x = 0,\ y = 0,\ z = 0,\ x = 1,\ y = 1,\ z = 1$, evaluate $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$.

(ii) Evaluate $\iint_S x^2 dy\, dz + y^2 dz\, dx + 2z\,(xy - x - y)\, dx\, dy$

where S is the surface of the cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$.

(Kumaun 2015)

3. (i) Evaluate $\iint_S [4xz\, dy\, dz - y^2\, dz\, dx + yz\, dx\, dy]$ where S is the surface of the cube bounded by the planes $x = 0, y = 0, z = 0, x = 1, y = 1$ and $z = 1$. **(Meerut 2005, 06B, 10B, 11)**

(ii) Apply Gauss's divergence theorem to evaluate

$$\iint_S [(x^3 - yz)\, dy\, dz - 2x^2 y\, dz\, dx + z\, dx\, dy]$$

over the surface of a cube bounded by the coordinate planes and the planes $x = y = z = a$. **(Rohilkhand 2011)**

4. (i) State divergence theorem of Gauss.

(ii) Use Gauss divergence theorem to show that

$$\iint_S \{(x^3 - yz)\,\mathbf{i} - 2x^2 y\,\mathbf{j} + 2\,\mathbf{k}\} \bullet \mathbf{n}\, dS = \frac{1}{3} a^5,$$

where S denotes the surface of the cube bounded by the planes $x = 0,\ x = a,\ y = 0,\ y = a,\ z = 0,\ z = a.$ **(Bundelkhand 2005, 06)**

5. Evaluate $\iint_S (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \bullet \mathbf{n}\, dS$ where S denotes the surface of the cube bounded by the planes $x = 0,\ y = 0,\ z = 0,\ x = a,\ y = a,\ z = a$ by the application of Gauss divergence theorem. Verify your answer by evaluating the integral directly. **(Garhwal 2003)**

6. (i) Evaluate by divergence theorem the integral

$$\iint_S xz^2\, dy\, dz + (x^2 y - z^3)\, dz\, dx + (2xy + y^2 z)\, dx\, dy,$$

where S is the entire surface of the hemispherical region bounded by $z = \sqrt{(a^2 - x^2 - y^2)}$ and $z = 0$.

(ii) Evaluate $\iint_S (y^2 z^2\,\mathbf{i} + z^2 x^2\,\mathbf{j} + z^2 y^2\,\mathbf{k}) \bullet \mathbf{n}\, dS$

where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane and bounded by this plane. **(Bundelkhand 2006)**

7. (i) If $\mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$, where a,b,c are constants, show that

$$\iint_S (\mathbf{n} \bullet \mathbf{F})\, dS = \frac{4\pi}{3}(a + b + c),$$

S being the surface of the sphere $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 1.$

(ii) If S is any closed surface enclosing a volume V and $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$, prove that $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = 6V.$ **(Rohilkhand 2009B)**

8. Verify the divergence theorem for $\mathbf{F} = 4x\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k}$ taken over the region bounded by the surfaces $x^2 + y^2 = 4,\ z = 0,\ z = 3.$ **(Garhwal 2002; Bundelkhand 08)**

9. Use Gauss divergence theorem to find $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$, where $\mathbf{F} = 2x^2 y\,\mathbf{i} - y^2\,\mathbf{j} + 4xz^2\,\mathbf{k}$ and S is the closed surface in the first octant bounded by $y^2 + z^2 = 9$ and $x = 2$.

10. If $\mathbf{F} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$, evaluate $\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane.

11. Evaluate $\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS$,

where $\mathbf{F} = (x^2 + y - 4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xz + z^2)\,\mathbf{k}$ and S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy-plane.

12. Compute

(i) $\iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{1/2}\, dS$, and

(ii) $\iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}\, dS$

over the ellipsoid $ax^2 + by^2 + cz^2 = 1$.

13. Evaluate $\iint_S (x^2 + y^2)\, dS$, where S is the surface of the cone $z^2 = 3(x^2 + y^2)$ bounded by $z = 0$ and $z = 3$.

14. Show that $\iint_S (x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}) \bullet \mathbf{n}\, dS$ vanishes where S denotes the surface of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

(Meerut 2005, 07; Kumaun 11, 13)

15. If **n** is the unit outward drawn normal to any closed surface S, show that

$$\iiint_V \text{div}\,\mathbf{n}\, dV = S.$$

Answers 2

2. (i) $\frac{3}{2}$ (ii) $\frac{1}{2}$

3. (i) $\frac{3}{2}$ (ii) $a^2\left(\frac{a^3}{3} + a\right)$

5. $3a^3$

6. (i) $\frac{2\pi a^5}{5}$ (ii) $\frac{\pi}{12}$

9. 180

10. 0

11. -4π

12. (i) $\frac{4}{3}\pi abc.$ (ii) $\frac{4\pi}{\sqrt{(abc)}}$

13. 9π

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5 Stoke's Theorem

Let S be a piecewise smooth open surface bounded by a piecewise smooth simple closed curve C. Let $\mathbf{F}(x, y, z)$ *be a continuous vector function which has continuous first partial derivatives in a region of space which contains S in its interior. Then*

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS = \iint_S (curl\ \mathbf{F}) \bullet d\mathbf{S}$$

where C is traversed in the positive direction. The direction of C is called positive if an observer, walking on the boundary of S in this direction, with his head pointing in the direction of outward drawn normal **n** *to S, has the surface on the left.*

(Meerut 2009; Bundelkhand 10)

Note: $\oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C \left(\mathbf{F} \bullet \frac{d\mathbf{r}}{ds}\right) ds = \oint_C (\mathbf{F} \bullet \mathbf{t})\, ds$, where **t** is unit tangent vector to C. Therefore $\mathbf{F} \bullet \mathbf{t}$ is the component of **F** in the direction of the tangent vector of C. Also $(\nabla \times \mathbf{F}) \bullet \mathbf{n}$ is the component of curl **F** in the direction of outward drawn normal vector **n** of S. Therefore in words Stoke's theorem may be stated as follows:

The line integral of the tangential component of vector **F** *taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of* **F** *taken over any surface S having C as its boundary.*

Cartesian equivalent of Stoke's theorem:

Let $\mathbf{F} = F_1\ \mathbf{i} + F_2\ \mathbf{j} + F_3\ \mathbf{k}$. Let outward drawn normal vector **n** of S make angles α, β, γ with positive directions of x, y, z axes.

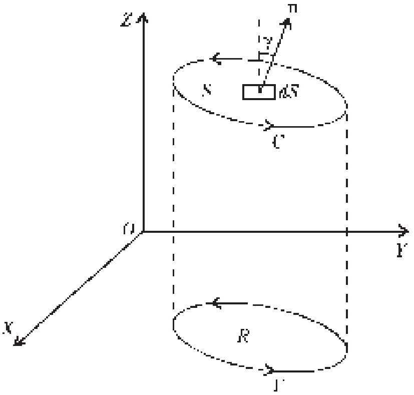
Then $\mathbf{n} = \cos\alpha\ \mathbf{i} + \cos\beta\ \mathbf{j} + \cos\gamma\ \mathbf{k}$.

Also
$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$
$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

$\therefore$
$$(\nabla \times \mathbf{F}) \bullet \mathbf{n} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma.$$

Also
$$\mathbf{F} \bullet d\mathbf{r} = (F_1\ \mathbf{i} + F_2\ \mathbf{j} + F_3\ \mathbf{k}) \bullet (dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k})$$
$$= F_1\ dx + F_2\ dy + F_3\ dz.$$

$\therefore$ Stoke's theorem can be written as
$$\oint_C F_1\ dx + F_2\ dy + F_3\ dz$$
$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] dS.$$



Proof of Stoke's theorem: Let S be a surface which is such that its projections on the xy, yz and zx planes are regions bounded by simple closed curves. Suppose S can be represented simultaneously in the forms $z = f(x, y)$, $y = g(x, z)$, $x = h(z, y)$ where f, g, h are continuous functions and have continuous first partial derivatives.

Consider the integral

$$\iint_S [\nabla \times (F_1\,\mathbf{i})] \bullet \mathbf{n}\, dS.$$

We have $\quad \nabla \times (F_1\,\mathbf{i}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix} = \dfrac{\partial F_1}{\partial z}\,\mathbf{j} - \dfrac{\partial F_1}{\partial y}\,\mathbf{k}.$

$$\therefore \quad [\nabla \times (F_1\,\mathbf{i})] \bullet \mathbf{n} = \left(\frac{\partial F_1}{\partial z}\,\mathbf{j} \bullet \mathbf{n} - \frac{\partial F_1}{\partial y}\,\mathbf{k} \bullet \mathbf{n}\right)$$

$$= \frac{\partial F_1}{\partial z} \cos\beta - \frac{\partial F_1}{\partial y} \cos\gamma.$$

$$\therefore \quad \iint_S [\nabla \times (F_1\,\mathbf{i})] \bullet \mathbf{n}\, dS = \iint_S \left(\frac{\partial F_1}{\partial z} \cos\beta - \frac{\partial F_1}{\partial y} \cos\gamma\right) dS.$$

We shall prove that

$$\iint_S \left(\frac{\partial F_1}{\partial z} \cos\beta - \frac{\partial F_1}{\partial y} \cos\gamma\right) dS = \oint_C F_1\, dx.$$

Let R be the orthogonal projection of S on the xy-plane and let Γ be its boundary which is oriented as shown in the figure. Using the representation $z = f(x, y)$ of S, we may write the line integral over C as a line integral over Γ. Thus

$$\oint_{\mathbf{C}} F_1(x, y, z)\, dx = \oint_{\Gamma} F_1[x, y, f(x, y)]\, dx$$

$$= \oint_{\Gamma} \{\mathrm{F}_1[x, y, f(x, y)]\, dx + 0\, dy\}$$

$$= -\iint_R \frac{\partial F_1}{\partial y}\, dx\, dy,$$

by Green's theorem in plane for the region R.

But $\quad \dfrac{\partial F_1[x, y, f(x, y)]}{\partial y} = \dfrac{\partial F_1(x, y, z)}{\partial y} + \dfrac{\partial F_1(x, y, z)}{\partial z} \dfrac{\partial f}{\partial y}.$

$[\because \; z = f(x, y)]$

$$\therefore \qquad \oint_C F_1\,(x, y, z)\,dx = -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\,dy \qquad \ldots(1)$$

Now the equation $z = f(x, y)$ of the surface S can be written as

$$\phi\,(x, y, z) \equiv z - f(x, y) = 0.$$

We have $$\text{grad }\phi = -\frac{\partial f}{\partial x}\,\mathbf{i} - \frac{\partial f}{\partial y}\,\mathbf{j} + \mathbf{k}.$$

Let $$|\text{grad }\phi| = a.$$

Since grad ϕ is normal to S, therefore, we get $\mathbf{n} = \pm \dfrac{\text{grad }\phi}{a}\cdot$

But the components of both $\mathbf{n}$ and grad ϕ in positive direction of z-axis are positive. Therefore

$$\mathbf{n} = + \frac{\text{grad }\phi}{a}$$

or $$\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k} = -\frac{1}{a}\frac{\partial f}{\partial x}\,\mathbf{i} - \frac{1}{a}\frac{\partial f}{\partial y}\,\mathbf{j} + \frac{1}{a}\,\mathbf{k}.$$

$$\therefore \qquad \cos\alpha = -\frac{1}{a}\frac{\partial f}{\partial x},\ \cos\beta = -\frac{1}{a}\frac{\partial f}{\partial y},\ \cos\gamma = \frac{1}{a}\cdot$$

Now $$dS = \frac{dx\,dy}{\cos\gamma} = a\,dx\,dy.$$

$$\begin{aligned}\therefore \qquad &\iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS \\ &= \iint_R \left[\frac{\partial F_1}{\partial z}\left(-\frac{1}{a}\frac{\partial f}{\partial y}\right) - \frac{\partial F_1}{\partial y}\frac{1}{a}\right] a\,dx\,dy \\ &= -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\,dy. \qquad \ldots(2)\end{aligned}$$

From (1) and (2), we get

$$\begin{aligned}\oint_C F_1\,dx &= \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS \\ &= \iint_S [\nabla \times (F_1\,\mathbf{i})] \bullet \mathbf{n}\,dS \qquad \ldots(3)\end{aligned}$$

Similarly, by projections on the other coordinate planes, we get

$$\oint_C F_2\,dy = \iint_S [\nabla \times (F_2\,\mathbf{j})] \bullet \mathbf{n}\,dS \qquad \ldots(4)$$

$$\oint_C F_3\,dz = \iint_S [\nabla \times (F_3\,\mathbf{k})] \bullet \mathbf{n}\,dS \qquad \ldots(5)$$

Adding (3), (4), (5), we get

$$\oint_C (F_1\,dx + F_2\,dy + F_3\,dz) = \iint_S [\nabla \times (F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k})] \bullet \mathbf{n}\,dS$$

or $$\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS.$$

If the surface S does not satisfy the restrictions imposed above, even then Stoke's theorem will be true provided S can be subdivided into surfaces $S_1, S_2, \ldots, S_k$ with boundaries $C_1, C_2, \ldots, C_k$ which do satisfy the restrictions. Stoke's theorem holds for each such surface. The sum of surface integrals over $S_1, S_2, \ldots, S_k$ will give us surface integral over S while the sum of the integrals over $C_1, C_2, \ldots, C_k$ will give us line integral over C.

Note: **Green's theorem in plane is a special case of Stoke's theorem.** If R is a region in the xy-plane bounded by a closed curve C, then in vector form Green's theorem in plane can be written as

$$\iint_R (\nabla \times \mathbf{F}) \bullet \mathbf{k}\, dR = \int_C \mathbf{F} \bullet d\mathbf{r}.$$

This is nothing but a special case of Stoke's theorem because here $\mathbf{k} = \mathbf{n}$ = outward drawn unit normal to the surface of region R.

Illustrative Examples

Example 9: *Prove that* $\oint_C \mathbf{r} \bullet d\mathbf{r} = 0.$ **(Meerut 2010)**

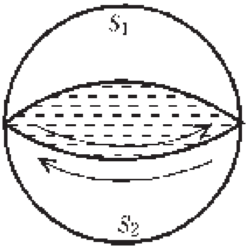
Solution: By Stoke's theorem $\oint_C \mathbf{r} \bullet d\mathbf{r} = \iint_S (\text{curl } \mathbf{r}) \bullet \mathbf{n}\, dS = 0$, since curl $\mathbf{r} = \mathbf{0}$.

Example 10: *By Stoke's theorem prove that div curl* $\mathbf{F} = 0$.

Solution: Let V be any volume enclosed by a closed surface.

Then by divergence theorem

$$\iiint_V \nabla \bullet (\text{curl } \mathbf{F})\, dV = \iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dS.$$



Divide the surface S into two portions S_1 and S_2 by a closed curve C. Then

$$\iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dS = \iint_{S_1} (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dS_1 + \iint_{S_2} (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dS_2 \,. \quad \ldots (1)$$

By Stoke's theorem right hand side of (1) is

$$= \oint_C \mathbf{F} \bullet d\mathbf{r} - \oint_C \mathbf{F} \bullet d\mathbf{r} = 0.$$

Negative sign has been taken in the second integral because the positive directions about the boundaries of the two surfaces are opposite.

$$\therefore \quad \iiint_V \nabla \bullet (\text{curl } \mathbf{F})\, dV = 0.$$

Now this equation is true for all volume elements V. Therefore we have

$$\nabla \bullet (\text{curl } \mathbf{F}) = 0 \quad \text{or} \quad \text{div curl } \mathbf{F} = 0.$$

Example 11: *Verify Stoke's theorem for* $\mathbf{F} = y\ \mathbf{i} + z\ \mathbf{j} + x\ \mathbf{k}$ *where S is the upper half surface of the sphere* $x^2 + y^2 + z^2 = 1$ *and C is its boundary.*

(Agra 2000, 06; Kanpur 09; Kumaun 07, 10, 13)

Solution: The boundary C of S is a circle in the xy-plane of radius unity and centre origin. The equations of the curve C are $x^2 + y^2 = 1,\ z = 0$. Suppose $x = \cos t,\ y = \sin t,\ z = 0, 0 \le t < 2\pi$ are parametric equation of C. Then

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C (y\ \mathbf{i} + z\ \mathbf{j} + x\ \mathbf{k}) \bullet (dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k})$$

$$= \oint_C (y\ dx + z\ dy + x\ dz)$$

$$= \oint_C y\ dx, \text{ since on } C,\ z = 0 \text{ and } dz = 0$$

$$= \int_0^{2\pi} \sin t \frac{dx}{dt}\ dt$$

$$= \int_0^{2\pi} -\sin^2 t\ dt$$

$$= -\frac{1}{2} \int_0^{2\pi} (1 - \cos 2t)\ dt$$

$$= -\frac{1}{2} \left[t - \frac{\sin 2t}{2} \right]_0^{2\pi}$$

$$= -\pi. \qquad \ldots(1)$$

Now let us evaluate $\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\ dS$. We have

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

If S_1 is the plane region bounded by the circle C, then by an application of divergence theorem, we have

$$\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\ dS = \iint_{S_1} \text{curl } \mathbf{F} \bullet \mathbf{k}\ dS$$

[See example 4 after article 4]

$$= \iint_{S_1} (-\mathbf{i} - \mathbf{j} - \mathbf{k}) \bullet \mathbf{k}\ dS$$

$$= \iint_{S_1} (-1)\ dS$$

$$= -\iint_{S_1} dS = -S_1.$$

But S_1 = area of a circle of radius $1 = \pi (1)^2 = \pi$.

$$\therefore \qquad \iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\ dS = -\pi. \qquad \ldots(2)$$

Hence from (1) and (2), the theorem is verified.

Example 12: *Verify Stoke's theorem for* $\mathbf{F} = (2x - y)\,\mathbf{i} - yz^2\,\mathbf{j} - y^2 z\,\mathbf{k}$*, where S is the upper half surface of the sphere* $x^2 + y^2 + z^2 = 1$ *and C is its boundary.*

(Kumaun 2003; Kanpur 10, 14; Avadh 09)

Solution: The boundary C of S is a circle in the xy-plane of radius unity and centre origin. Suppose $x = \cos t,\ y = \sin t,\ z = 0, 0 \le t < 2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C [(2x - y)\mathbf{i} - yz^2\,\mathbf{j} - y^2 z\,\mathbf{k}] \bullet (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C [(2x - y)\,dx - yz^2\,dy - y^2 z\,dz]$$

$$= \oint_C (2x - y)\,dx, \text{ since } z = 0 \text{ and } dz = 0$$

$$= \int_0^{2\pi} (2\cos t - \sin t)\,\frac{dx}{dt}\,dt$$

$$= -\int_0^{2\pi} (2\cos t - \sin t)\sin t\,dt$$

$$= -\int_0^{2\pi} \left[\sin 2t - \frac{1}{2}(1 - \cos 2t)\right] dt$$

$$= -\left[-\frac{\cos 2t}{2} - \frac{1}{2}t + \frac{1}{2}\,\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= -\left[\left(-\frac{1}{2} + \frac{1}{2}\right) - \frac{1}{2}(\pi - 0) + \frac{1}{4}(0 - 0)\right]$$

$$= \pi. \qquad \ldots(1)$$

And
$$(\nabla \times \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= (-2yz + 2yz)\,\mathbf{i} - (0 - 0)\,\mathbf{j} + (0 + 1)\,\mathbf{k}$$

$$= \mathbf{k}.$$

Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface.

∴ by an application of Gauss divergence theorem, we have

$$\iint_{S'} \text{curl } \mathbf{F} \bullet \mathbf{n}\,dS = 0 \qquad \text{[See example 4(i) after article 4]}$$

or
$$\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\,dS + \iint_{S_1} \text{curl } \mathbf{F} \bullet \mathbf{n}\,dS = 0$$

$[\because\ S' \text{ consists of } S \text{ and } S_1]$

or
$$\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\,dS - \iint_{S_1} \text{curl } \mathbf{F} \bullet \mathbf{k}\,dS = 0$$

$[\because\ \text{on } S_1,\ \mathbf{n} = -\mathbf{k}\ ]$

or
$$\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\,dS = \iint_{S_1} \text{curl } \mathbf{F} \bullet \mathbf{k}\,dS.$$

$$\therefore \qquad \iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\, dS = \iint_{S_1} \text{curl } \mathbf{F} \bullet \mathbf{k}\, dS$$

$$= \iint_{S_1} \mathbf{k} \bullet \mathbf{k}\, dS$$

$$= \iint_{S_1} dS = S_1 = \pi. \qquad \ldots(2)$$

Note that S_1 = area of a circle of radius $1 = \pi\,(1)^2 = \pi$.

Hence from (1) and (2) Stoke's theorem is verified.

Example 13: *Verify Stoke's theorem for* $\mathbf{F} = (x^2 + y^2)\,\mathbf{i} - 2xy\mathbf{j}$ *taken round the rectangle bounded by* $x = \pm a, \; y = 0, \; y = b.$ **(Bundelkhand 2007; Agra 08; Kumaun 15)**

Solution: We have curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$

$$= (-2y - 2y)\,\mathbf{k} = -4y\,\mathbf{k}.$$

Also $\qquad \mathbf{n} = \mathbf{k}.$

$$\therefore \qquad \iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dS$$

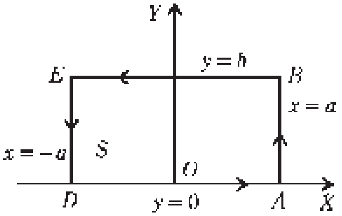
$$= \int_{y=0}^{b} \int_{x=-a}^{a} (-4y\,\mathbf{k}) \bullet \mathbf{k}\, dx\, dy$$

$$= -4 \int_{y=0}^{b} \int_{x=-a}^{a} y\, dx\, dy$$

$$= -4 \int_{y=0}^{b} [xy]_{x=-a}^{a}\, dy$$

$$= -4 \int_{y=0}^{b} 2ay\, dy$$

$$= -4\,[ay^2]_0^b = -4ab^2.$$



Also $\qquad \oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C [(x^2 + y^2)\,\mathbf{i} - 2xy\,\mathbf{j}] \bullet (dx\,\mathbf{i} + dy\,\mathbf{j})$

$$= \oint_C [(x^2 + y^2)\, dx - 2xy\, dy]$$

$$= \int_{DA} [(x^2 + y^2)\, dx - 2xy\, dy] + \int_{AB} + \int_{BE} + \int_{ED}.$$

Along DA, $y = 0$ and $dy = 0$. Along AB, $x = a$ and $dx = 0$.
Along BE, $y = b$ and $dy = 0$. Along ED, $x = -a$ and $dx = 0$.

$$\therefore \qquad \oint_C \mathbf{F} \bullet d\mathbf{r} = \int_{x=-a}^{a} x^2\, dx + \int_{y=0}^{b} -2ay\, dy$$

$$+ \int_{x=a}^{-a} (x^2 + b^2)\, dx + \int_{y=b}^{0} 2ay\, dy$$

$$= \int_{-a}^{a} x^2\, dx - \int_{-a}^{a} (x^2 + b^2)\, dx - 4a \int_0^b y\, dy$$

$$= -\int_{-a}^{a} x^2 \, dx - 4a \int_0^b y \, dy$$

$$= -2ab^2 - 4a\left[\frac{y^2}{2}\right]_0^b = -4ab^2.$$

Thus $\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n} \, dS.$

Hence the theorem is verified.

Example 14: *Evaluate* $\oint_C \mathbf{F} \bullet d\mathbf{r}$ *by Stoke's theorem where*

$$\mathbf{F} = y^2 \, \mathbf{i} + x^2 \, \mathbf{j} - (x+z) \, \mathbf{k}$$

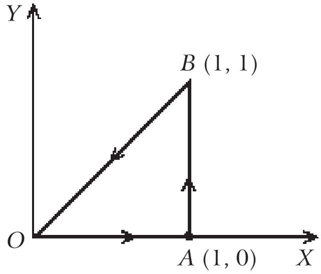
and C is the boundary of the triangle with vertices at $(0, 0, 0), (1, 0, 0), (1, 1, 0)$.

(Avadh 2013)

Solution: We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & x^2 & (x+z) \end{vmatrix}$$

$$= 0 \, \mathbf{i} + \mathbf{j} + 2(x-y) \, \mathbf{k}.$$



Also we note that z co-ordinate of each vertex of the triangle is zero. Therefore the triangle lies in the xy-plane. So $\mathbf{n} = \mathbf{k}$.

$$\therefore \quad \text{Curl } \mathbf{F} \bullet \mathbf{n} = [\,\mathbf{j} + 2(x-y)\mathbf{k}] \bullet \mathbf{k}$$

$$= 2(x-y).$$

In the figure, we have only considered the xy plane.

The equation of the line OB is $y = x$.

By Stoke's theorem

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n} \, dS$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} 2(x-y) \, dx \, dy$$

$$= 2\int_{x=0}^{1} \left[x\,y - \frac{y^2}{2}\right]_{y=0}^{x} dx$$

$$= 2\int_0^1 \left[x^2 - \frac{x^2}{2}\right] dx$$

$$= 2\int_0^1 \frac{x^2}{2} \, dx$$

$$= \int_0^1 x^2 \, dx = \frac{1}{3}.$$

Comprehensive Exercise 3

1. (i) State Stoke's theorem.

(ii) By Stoke's theorem prove that curl grad $\phi = \mathbf{0}$. **(Kumaun 2014)**

(iii) Verify Stoke's theorem for the function $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\,\mathbf{k}$ where curve is the unit circle in the xy-plane bounding the hemisphere $z = \sqrt{(1 - x^2 - y^2)}$. **(Garhwal 2003; Agra 07)**

2. Verify Stoke's theorem for the vector $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ taken over the half of the sphere $x^2 + y^2 + z^2 = a^2$ lying above the xy-plane.

3. (i) Verify Stoke's theorem for the function $\mathbf{F} = x^2\mathbf{i} + xy\,\mathbf{j}$ integrated along the rectangle, in the plane $z = 0$, whose sides are along the lines $x = 0,\ y = 0,\ x = a$ and $y = b$.

(ii) Verify Stoke's theorem for the function $\mathbf{F} = (x^2 - y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$ in the rectangular region in the xy-plane bounded by the lines $x = 0,\ x = a,\ y = 0$ and $y = b$. **(Kanpur 2008)**

(iii) Verify Stoke's theorem for the function $\mathbf{F} = x^2\mathbf{i} + xy\,\mathbf{j}$, integrated round the square, in the plane $z = 0$, whose sides are along the lines $x = 0,\ y = 0,\ x = a,\ y = a$. **(Agra 2002)**

4. Verify Stoke's theorem for the vector $\mathbf{A} = 3y\,\mathbf{i} - xz\,\mathbf{j} + yz^2\,\mathbf{k}$, where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and C is its boundary.

5. (i) By converting into a line integral evaluate $\iint_S (\nabla \times \mathbf{A}) \bullet \mathbf{n}\, dS$, where $\mathbf{A} = (x - z)\,\mathbf{i} + (x^3 + yz)\,\mathbf{j} - 3xy^2\,\mathbf{k}$ and S is the surface of the cone $z = 2 - \sqrt{(x^2 + y^2)}$ above the xy-plane.

(ii) By converting into a line integral evaluate $\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS$ where $\mathbf{F} = (x^2 + y - 4)\,\mathbf{i} + 3\,xy\,\mathbf{j} + (2xy + z^2)\,\mathbf{k}$ and S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy-plane.

6. (i) Evaluate by Stoke's theorem $\oint_C (e^x\,dx + 2y\,dy - dz)$ where C is the curve $x^2 + y^2 = 4,\ z = 2$. **(Garhwal 2001; Meerut 09B)**

(ii) Evaluate by Stoke's theorem $\oint_C (yz\,dx + xz\,dy + xy\,dz)$ where C is the curve $x^2 + y^2 = 1,\ z = y^2$. **(Meerut 2006B)**

7. (i) Evaluate $\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS$, where $\mathbf{F} = (y - z + 2)\,\mathbf{i} + (yz + 4)\,\mathbf{j} - xz\,\mathbf{k}$ and S is the surface of the cube $x = y = z = 0$, $x = y = z = 2$ above the xy-plane.

(ii) Evaluate by Stoke's theorem $\oint_C (\sin z\, dx - \cos x\, dy + \sin y\, dz)$ where C is the boundary of the rectangle $0 \le x \le \pi, 0 \le y \le 1, z = 3$.

(Meerut 2009)

8. If $\mathbf{f} = \nabla \phi$ and $g = \nabla \psi$ are two vector point functions, such that $\nabla^2 \phi = 0$, $\nabla^2 \psi = 0$, show that

$$\iint_S (\mathbf{g} \bullet \nabla)\, \mathbf{f} \bullet d\mathbf{S} = \int_C (\mathbf{f} \times \mathbf{g}) \bullet d\mathbf{r} + \iint_S (\mathbf{f} \bullet \nabla)\mathbf{g} \bullet d\mathbf{S}.$$

9. Prove that a necessary and sufficient condition that $\oint_C \mathbf{F} \bullet d\mathbf{r} = 0$ for every closed curve C lying in a simply connected region R is that $\nabla \times \mathbf{F} = \mathbf{0}$ identically.

10. Apply Stoke's theorem to prove that

$$\int_C (y\, dx + z\, dy + x\, dz) = -2\sqrt{2}\,\pi a^2$$

where C is the curve given by $x^2 + y^2 + z^2 - 2ax - 2ay = 0, x + y = 2a$ and begins at the point $(2a, 0, 0)$ and goes at first below the z-plane.

(Meerut 2005, 06B)

11. Use Stoke's theorem to evaluate $\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS$, where $\mathbf{F} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$ and S is the surface of sphere $x^2 + y^2 + z^2 = a^2$, above the xy-plane.

(Kumaun 2012)

Answers 3

5. (i) 12π (ii) -4π **6.** (i) 0 (ii) 0

7. (i) -4 (ii) 2

11. 0

6 Line Integrals Independent of Path

Let $\mathbf{F}(x, y, z) = f(x, y, z)\,\mathbf{i} + g(x, y, z)\,\mathbf{j} + h(x, y, z)\,\mathbf{k}$ be a vector point function defined and continuous in a region R of space. Let P and Q be two points in R and let C be a path joining P to Q. Then

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int (f\, dx + g\, dy + h\, dz) \qquad \dots(1)$$

is called the line integral of **F** along C. In general the value of this line integral depends not only on the end points P and Q of the path C but also on C.

In other words, if we integrate from P to Q along different paths, we shall, in general, get different values of the integral. *The line integral* (1) *is said to be* **independent of path** *in R, if for every pair of end points P and Q in R the value of the integral is the same for all paths C in R starting from P and ending at Q.*

In this case the value of this line integral will depend on the choice of P and Q and not on the choice of the path joining P to Q.

Definition: *The expression* $f\,dx + g\,dy + h\,dz$ *is said to be an exact differential if there exists a single valued scalar point function* $\phi\,(x, y, z)$, *having continuous first partial derivatives such that* $d\phi = f\,dx + g\,dy + h\,dz$.

It can be easily seen that $f\,dx + g\,dy + h\,dz$ is an exact differential if and only if the vector function $\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$ is the gradient of a single valued scalar function $\phi\,(x, y, z)$.

Because $\mathbf{F} = \text{grad}\ \phi$

if, and only if $f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k} = \dfrac{\partial \phi}{\partial x}\,\mathbf{i} + \dfrac{\partial \phi}{\partial y}\,\mathbf{j} + \dfrac{\partial \phi}{\partial z}\,\mathbf{k}$

if, and only if $f = \dfrac{\partial \phi}{\partial x}, \quad g = \dfrac{\partial \phi}{\partial y}, \quad h = \dfrac{\partial \phi}{\partial z}$

if, and only if $f\,dx + g\,dy + h\,dz = \dfrac{\partial \phi}{\partial x}\,dx + \dfrac{\partial \phi}{\partial y}\,dy + \dfrac{\partial \phi}{\partial z}\,dz$

if, and only if $f\,dx + g\,dy + h\,dz = d\,\phi$.

Thus $\mathbf{F} = \text{grad}\ \phi$ if, and only if $f\,dx + g\,dy + h\,dz$ is an exact differential $d\phi$.

Theorem 1: *Let f (x, y, z), g (x, y, z) and h (x, y, z) be continuous in a region R of space. Then the line integral*

$$\int (f\,dx + g\,dy + h\,dz)$$

is independent of path in R if and only if the differential form under the integral sign is exact in R.

Or

Let $\mathbf{F}\,(x, y, z)$ *be continuous in region R of space. Then the line integral* $\int_C \mathbf{F} \bullet d\mathbf{r}$

is independent of the path C in R joining P and Q if and only if $\mathbf{F} = \text{grad}\ \phi$ *where* $\phi\,(x, y, z)$ *is a single-valued scalar function having continuous first partial derivatives in R.*

Proof: Suppose $\mathbf{F} = \text{grad}\ \phi$ in R. Let P and Q be any two points in R and let C be any path from P to Q in R.

Then $$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \nabla\phi \bullet d\mathbf{r}$$

$$
\begin{aligned}
&= \int_C \left(\frac{\partial \phi}{\partial x}\,\mathbf{i} + \frac{\partial \phi}{\partial y}\,\mathbf{j} + \frac{\partial \phi}{\partial z}\,\mathbf{k}\right) \bullet (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})\\
&= \int_C \left(\frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz\right)\\
&= \int_C d\phi\\
&= \int_P^Q d\phi\\
&= [\phi]_P^Q\\
&= \phi(Q) - \phi(P).
\end{aligned}
$$

Thus the line integral depends only on points P and Q and not on the path joining them. This is true, of course, only if $\phi(x, y, z)$ is single valued at all points P and Q.

Conversely, suppose the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ is independent of the path C joining any two points P and Q in R. Let P be a fixed point (x_0, y_0, z_0) in R and let Q be any point (x, y, z) in R.

Let $$\phi(x, y, z) = \int_{(x_0, y_0, z_0)}^{(x, y, z)} \mathbf{F} \bullet d\mathbf{r}$$

$$= \int_{x_0, y_0, z_0}^{(x, y, z)} \left(\mathbf{F} \bullet \frac{d\mathbf{r}}{ds}\right) ds.$$

Differentiating both sides with respect to s, we get

$$\frac{d\phi}{ds} = \mathbf{F} \bullet \frac{d\mathbf{r}}{ds}.$$

But $$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds}$$

$$= \left(\frac{\partial \phi}{\partial x}\,\mathbf{i} + \frac{\partial \phi}{\partial y}\,\mathbf{j} + \frac{\partial \phi}{\partial z}\,\mathbf{k}\right) \bullet \left(\frac{dx}{ds}\,\mathbf{i} + \frac{dy}{ds}\,\mathbf{j} + \frac{dz}{ds}\,\mathbf{k}\right)$$

$$= \nabla \phi \bullet \frac{d\mathbf{r}}{ds}.$$

∴ $$\mathbf{F} \bullet \frac{d\mathbf{r}}{ds} = \nabla \phi \bullet \frac{d\mathbf{r}}{ds}$$

or $$(\nabla \phi - \mathbf{F}) \bullet \frac{d\mathbf{r}}{ds} = 0.$$

Now this result is true irrespective of the path joining P to Q *i.e.* this result is true irrespective of the direction of $\frac{d\mathbf{r}}{ds}$ which is tangent vector to C. Therefore we must have

$$\nabla\phi - \mathbf{F} = \mathbf{0}$$

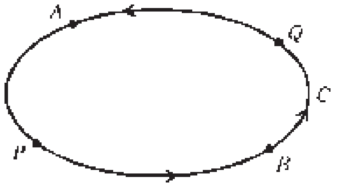
i.e., $$\nabla \phi = \mathbf{F}.$$

This completes the proof of the theorem.

Definition: *A vector field* $\mathbf{F}(x, y, z)$ *defined and continuous in a region R of space is said to be a conservative vector field if the line integral* $\int_C \mathbf{F} \bullet d\mathbf{r}$ *is independent of the path C in R joining P and Q where P and Q are any two points in R.*

By theorem 1, vector field $\mathbf{F}(x, y, z)$ is conservative if and only if $\mathbf{F} = \nabla \phi$ where $\phi(x, y, z)$ is a single valued scalar function having continuous first partial derivatives in *R*. The function $\phi(x, y, z)$ is called the **scalar potential** *of the vector field* **F**.

Theorem 2: *Let* $\mathbf{F}(x, y, z)$ *be a vector function defined and continuous in a region R of space. Then the line integral* $\int_P^Q \mathbf{F} \bullet d\mathbf{r}$ *is independent of the path joining any two points P and Q in R if and only if* $\oint_{\mathbf{C}} \mathbf{F} \bullet \mathbf{dr} = 0$ *for every simple closed path in R.*



Proof: Let *C* be any simple closed path in *R* and let the line integral be independent of path in *R*. Take two points *P* and *Q* on *C* and subdivide *C* into two arcs *PBQ* and *QAP*. Then

$$\begin{aligned}\oint_{\mathbf{C}} \mathbf{F} \bullet d\mathbf{r} &= \int_{PBQAP} \mathbf{F} \bullet d\mathbf{r} \\ &= \int_{PBQ} \mathbf{F} \bullet d\mathbf{r} + \int_{QAP} \mathbf{F} \bullet d\mathbf{r} \\ &= \int_{PBQ} \mathbf{F} \bullet d\mathbf{r} - \int_{PAQ} \mathbf{F} \bullet d\mathbf{r} \\ &= 0,\end{aligned}$$

since the integral from *P* to *Q* along a path through *B* is equal to the integral from *P* to *Q* along a path through *A*.

Conversely, suppose that the integral under consideration is zero on every simple closed path in *R*. Let *P* and *Q* be any two points in *R* which join *P* to *Q* and do not cross. Then

$$\begin{aligned}\oint_{PBQAP} \mathbf{F} \bullet d\mathbf{r} &= \int_{PBQ} \mathbf{F} \bullet d\mathbf{r} + \int_{QAP} \mathbf{F} \bullet d\mathbf{r} \\ &= \int_{PBQ} \mathbf{F} \bullet d\mathbf{r} - \int_{PAQ} \mathbf{F} \bullet d\mathbf{r}.\end{aligned}$$

But as given, we have

$$\oint_{PBQAP} \mathbf{F} \bullet d\mathbf{r} = 0.$$

$$\therefore \qquad \int_{PBQ} \mathbf{F} \bullet d\mathbf{r} - \int_{PAQ} \mathbf{F} \bullet d\mathbf{r} = 0$$

or

$$\int_{PBQ} \mathbf{F} \bullet d\mathbf{r} = \int_{PAQ} \mathbf{F} \bullet d\mathbf{r}.$$

This completes the proof of the theorem.

Theorem 3: *Let* $\mathbf{F}(x, y, z) = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$ *be a continuous vector function having continuous first partial derivatives in a region R of space. If* $\int f\,dx + g\,dy + h\,dz$ *is independent of path in R and consequently* $f\,dx + g\,dy + h\,dz$ *is an exact differential in R, then curl* $\mathbf{F} = \mathbf{0}$ *everywhere in R. Conversely, if R is simply connected and curl* $\mathbf{F} = \mathbf{0}$ *everywhere in R, then* $f\,dx + g\,dy + h\,dz$ *is an exact differential in R or* $\int f\,dx + g\,dy + h\,dz$ *is independent of path in R.*

Proof: Suppose $\int (f\,dx + g\,dy + h\,dz)$ is independent of path in R. Then $f\,dx + g\,dy + h\,dz$ is an exact differential in R. Therefore

$$\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k} = \text{grad } \phi.$$

$$\therefore \qquad \text{curl } \mathbf{F} = \text{curl (grad } \phi) = \mathbf{0}.$$

Converse. Suppose R is simply connected and curl $\mathbf{F} = \mathbf{0}$ everywhere in R. Let C be any simple closed path in R. Since R is simply connected, therefore we can find a surface S in R having C as its boundary. Therefore by Stoke's theorem

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dS = 0.$$

Thus $\oint_C \mathbf{F} \bullet d\mathbf{r}$ is zero for every simple closed path C in R.

Therefore $\int \mathbf{F} \bullet d\mathbf{r}$ is independent of path in R.

Therefore $\mathbf{F} = \nabla \phi$ and consequently $f\,dx + g\,dy + h\,dz$ is an exact differential $d\phi$.

Note: The assumption that R be simply connected is essential and cannot be omitted. It is obvious from the following illustration.

Illustration: Let

$$\mathbf{F} = -\frac{y}{x^2 + y^2}\,\mathbf{i} + \frac{x}{x^2 + y^2}\,\mathbf{j}.$$

Here $\mathbf{F}$ is not defined at origin. In every region R of the xy-plane not containing the origin, we have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\dfrac{y}{x^2 + y^2} & \dfrac{x}{x^2 + y^2} & 0 \end{vmatrix}$$

$$= 0\mathbf{i} + 0\mathbf{j} + \left\{ \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right) \right\} \mathbf{k}$$

$$= \left\{ \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} \right\} \mathbf{k}$$

$$= 0\ \mathbf{k}$$

$$= \mathbf{0}.$$

Suppose R is simply connected. For example let R be the region enclosed by a simple closed curve C not enclosing the origin. Then

$$\oint_{\mathbf{C}} \mathbf{F} \bullet d\mathbf{r} = \oint_{\mathbf{C}} \left(-\frac{y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy \right)$$

$$= \int\int_R \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2} \right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2 + y^2} \right) \right] dx\, dy,$$

by Green's theorem in plane

$$= 0.$$

Suppose R is not simply connected. Let R be the region of the xy-plane contained between concentric circles of radii $\frac{1}{2}$ and $\frac{3}{2}$ and having centre at origin. Obviously R is not simply connected. We have $z = 0$, everywhere in R. Let C be a closed curve in R. The parametric equations of C can be taken as $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t < 2\pi$.

We have $$\oint_{\mathbf{C}} \mathbf{F} \bullet d\mathbf{r} = \int_C \left(-\frac{y}{x^2 + y^2}\, dx + \frac{y}{x^2 + y^2}\, dy \right)$$

$$= \int_{t=0}^{2\pi} \left[-\frac{\sin t}{\cos^2 t + \sin^2 t} \frac{dx}{dt} + \frac{\cos t}{\cos^2 t + \sin^2 t} \frac{dy}{dt} \right] dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$$

$$= 2\,\pi.$$

Thus we see that

$$\oint_{\mathbf{C}} \mathbf{F} \bullet d\mathbf{r} \neq 0.$$

Definition: Irrotational vector field: *A vector field* **F** *is said to be irrotational if curl* **F = 0.**

We see that an irrotational field **F** is characterized by any one of the three conditions :

(i) $\mathbf{F} = \nabla\, \phi$,

(ii) $\nabla \times \mathbf{F} = \mathbf{0}$,

(iii) $\oint_{\mathbf{C}} \mathbf{F} \bullet d\mathbf{r} = 0$ for every closed path.

Any one of these conditions implies the other two.

Illustrative Examples

Example 15: *Are the following forms exact ?*

(i) $e^y \, dx + e^x \, dy + e^z \, dz$.

(ii) $yz \, dx + xz \, dy + xy \, dz$.

Solution:

(i) Here $\mathbf{F} = e^y \, \mathbf{i} + e^x \, \mathbf{j} + e^z \, \mathbf{k}$.

We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^y & e^x & e^z \end{vmatrix}$$

$$= 0 \, \mathbf{i} + 0 \, \mathbf{j} + (e^x - e^y) \, \mathbf{k}.$$

Since curl $\mathbf{F} \neq \mathbf{0}$, therefore the given form is not exact.

(ii) Here $\mathbf{F} = yz \, \mathbf{i} + xz \, \mathbf{j} + xy \, \mathbf{k}$.

We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= (x - x)\mathbf{i} - (y - y)\mathbf{j} + (z - z)\mathbf{k}$$

$$= \mathbf{0}.$$

Since curl $\mathbf{F} = \mathbf{0}$, therefore the given form is exact.

Example 16: *In each of following cases show that the given differential form is exact and find a function ϕ such that the form equals $d\phi$:*

(i) $\cos x \, dx - 2yz \, dy - y^2 \, dz$.

(ii) $(z^2 - 2xy) \, dx - x^2 \, dy + 2xz \, dz$.

Solution: (i) Here $\mathbf{F} = \cos x \, \mathbf{i} - 2yz \, \mathbf{j} - y^2 \, \mathbf{k}$.

We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \cos x & -2yz & -y^2 \end{vmatrix}$$

$$= (-2y + 2y) \, \mathbf{i} + 0 \, \mathbf{j} + 0 \, \mathbf{k} = \mathbf{0}.$$

$\therefore$ the given form is exact.

Let $\mathbf{F} = \nabla\,\phi$,

or $\cos x\,\mathbf{i} - 2yz\,\mathbf{j} - y^2\,\mathbf{k} = \dfrac{\partial\phi}{\partial x}\,\mathbf{i} + \dfrac{\partial\phi}{\partial y}\,\mathbf{j} + \dfrac{\partial\phi}{\partial z}\,\mathbf{k}$.

Then $\dfrac{\partial\phi}{\partial x} = \cos x$ whence $\phi = \sin x + f_1\,(y, z)$ ... (1)

$\dfrac{\partial\phi}{\partial y} = -2yz$ whence $\phi = -y^2 z + f_2\,(x, z)$ ... (2)

$\dfrac{\partial\phi}{\partial z} = -y^2$ whence $\phi = -y^2 z + f_3\,(x, y)$. ... (3)

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1\,(y, z) = -y^2 z,\ f_2\,(x, z) = \sin x,\ f_3\,(x, y) = \sin x.$

∴ $\phi = \sin x - y^2 z$ to which may be added any constant.

∴ $\phi = \sin x - y^2 z + C$.

(ii) Here $\mathbf{F} = (z^2 - 2xy)\,\mathbf{i} - x^2\,\mathbf{j} + 2xz\,\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 - 2xy & -x^2 & 2xz \end{vmatrix}$$

$$= 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{0}.$$

∴ the given form is exact.

Let $\mathbf{F} = \nabla\,\phi$

or $(z^2 - 2xy)\,\mathbf{i} - x^2\,\mathbf{j} + 2xz\,\mathbf{k} = \dfrac{\partial\phi}{\partial x}\,\mathbf{i} + \dfrac{\partial\phi}{\partial y}\,\mathbf{j} + \dfrac{\partial\phi}{\partial z}\,\mathbf{k}$.

Then $\dfrac{\partial\phi}{\partial x} = z^2 - 2xy$ whence $\phi = z^2 x - x^2 y + f_1\,(y, z)$ ... (1)

$\dfrac{\partial\phi}{\partial y} = -x^2$ whence $\phi = -x^2 y + f_2\,(x, z)$ ... (2)

$\dfrac{\partial\phi}{\partial z} = 2xz$ whence $\phi = xz^2 + f_3\,(x, y)$. ... (3)

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1\,(y, z) = 0,\ f_2\,(x, z) = xz^2,\ f_3\,(x, y) = -x^2 y.$

∴ $\phi = z^2 x - x^2 y$ to which may be added any constant.

∴ $\phi = z^2 x - x^2 y + C$.

Example 17: *Show that the vector field* $\mathbf{F}$ *given by*

$$\mathbf{F} = (x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k}$$

is irrotational. Find a scalar ϕ *such that* $\mathbf{F} = \nabla\phi$.

Solution: We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= (-x + x)\,\mathbf{i} - (-y + y)\,\mathbf{j} + (-z + z)\,\mathbf{k} = \mathbf{0}.$$

∴ The vector field **F** is irrotational.

Let $\mathbf{F} = \nabla\phi$

or $(x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k} = \dfrac{\partial\phi}{\partial x}\,\mathbf{i} + \dfrac{\partial\phi}{\partial y}\,\mathbf{j} + \dfrac{\partial\phi}{\partial z}\,\mathbf{k}.$

Then $\dfrac{\partial\phi}{\partial x} = x^2 - yz$ whence $\phi = \dfrac{x^3}{3} - xyz + f_1(y, z)$ ... (1)

$\dfrac{\partial\phi}{\partial y} = y^2 - zx$ whence $\phi = \dfrac{y^3}{3} - xyz + f_2(x, z)$ ... (2)

$\dfrac{\partial\phi}{\partial z} = z^2 - xy$ whence $\phi = \dfrac{z^3}{3} - xyz + f_3(x, y).$ ... (3)

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z) = \frac{y^3}{3} + \frac{z^3}{3},$$

$$f_2(x, z) = \frac{x^3 + z^3}{3},$$

$$f_3(x, y) = \frac{x^3 + y^3}{3}.$$

Therefore $\phi = \dfrac{x^3 + y^3 + z^3}{3} - xyz + C.$

7 Physical Interpretation of Divergence and Curl

Physical interpretation of divergence: *Suppose that there is a fluid motion whose velocity at any point is* **v** *(x, y, z). Then the loss of fluid per unit volume per unit time in a small parallelopiped having centre at P (x, y, z) and edges parallel to the co-ordinate axes and having lengths δx, δy, δz respectively, is given approximately by*

$$\text{div } \mathbf{v} = \nabla \bullet \mathbf{v}.$$

Let $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}.$

x-component of velocity **v** at $P = v_1(x, y, z)$.

∴ x-component of **v** at centre of face $AFED$ which is perpendicular to x-axis and is nearer to origin

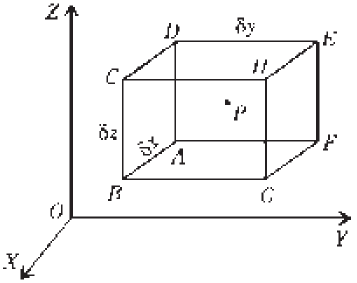
$$= v_1 \left(x - \frac{\delta x}{2}, y, z \right)$$

$$= v_1\ (x, y, z) - \frac{\delta x}{2} \frac{\partial v_1}{\partial x} + \ldots$$

by Taylor's theorem

$$= v_1\ (x, y, z) - \frac{\delta x}{2} \frac{\partial v_1}{\partial x}$$

approximately.



Similarly x-component of **v** at centre of opposite face *GHCB*

$$= v_1 + \frac{\delta x}{2} \frac{\partial v_1}{\partial x} \text{ approximately.}$$

∴ volume of fluid entering the parallelopiped across *AFED* per unit time

$$= \left(v_1 - \frac{\delta x}{2} \frac{\partial v_1}{\partial x} \right) \delta y\ \delta z.$$

Also volume of fluid going out the parallelopiped across *GHCB* per unit time

$$= \left(v_1 + \frac{\delta x}{2} \frac{\partial v_1}{\partial x} \right) \delta y\ \delta z.$$

∴ loss in volume per unit time in the direction of x-axis

$$= \left(v_1 + \frac{\delta x}{2} \frac{\partial v_1}{\partial x} \right) \delta y\ \delta z - \left(v_1 - \frac{\delta x}{2} \frac{\partial v_1}{\partial x} \right) \delta y\ \delta z$$

$$= \frac{\partial v_1}{\partial x}\ \delta x\ \delta y\ \delta z.$$

Similarly, loss in volume per unit time in y direction

$$= \frac{\partial v_2}{\partial y}\ \delta x\ \delta y\ \delta z,$$

and loss in volume per unit time in z direction

$$= \frac{\partial v_3}{\partial z}\ \delta x\ \delta y\ \delta z.$$

∴ total loss of the fluid per unit volume per unit time symbol ·

$$= \frac{\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) \delta x\ \delta y\ \delta z}{\delta x\ \delta y\ \delta z}$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$= \nabla \bullet \mathbf{v} = \text{div }\ \mathbf{v}.$$

Physical interpretation of curl: Let S be a circular disc of small radius r and centre P bounded by the circle C. Let $\mathbf{F}(x, y, z)$ be a continuously differentiable vector function in S. Then by Stoke's theorem

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n} \, dS$$

$$= \overline{(\text{curl } \mathbf{F}) \bullet \mathbf{n}} \iint_S dS,$$

by mean value theorem of integral calculus where $\overline{(\text{curl } \mathbf{F}) \bullet \mathbf{n}}$ is some value intermediate between the maximum and minimum values of $(\text{curl } \mathbf{F}) \bullet \mathbf{n}$ over S.

$$\therefore \qquad \int_C \mathbf{F} \bullet d\mathbf{r} = \overline{(\text{curl } \mathbf{F}) \bullet \mathbf{n}} \; S.$$

$$\therefore \qquad \overline{(\text{curl } \mathbf{F}) \bullet \mathbf{n}} = \frac{\left[\oint_C \mathbf{F} \bullet d\mathbf{r}\right]}{S}.$$

Taking limit as $r \to 0$, we get at P,

$$(\text{curl } \mathbf{F}) \bullet \mathbf{n} = \lim_{r \to 0} \frac{\oint_C \mathbf{F} \bullet d\mathbf{r}}{S}.$$

Now $(\text{curl } \mathbf{F}) \bullet \mathbf{n}$ is normal component of curl $\mathbf{F}$ at P and $\oint_C \mathbf{F} \bullet d\mathbf{r}$ is circulation of $\mathbf{F}$ about C. Therefore the normal component of the curl can be interpreted physically as the limit of the circulation per unit area.

Comprehensive Exercise 4

1. Are the following forms exact ?
   (i) $x\,dx - y\,dy + z\,dz$.
   (ii) $y^2 z^3 \, dx + 2xyz^3 \, dy + 3xy^2 z^2 \, dz$.
2. In each of following cases show that the given differential form is exact and find a function ϕ such that the form equals $d\phi$:
   (i) $x\,dx - y\,dy - z\,dz$.
   (ii) $dx + z\,dy + y\,dz$.
3. (i) Show that
   $(y^2 z^3 \cos x - 4x^3 z)\,dx + 2z^3 y \sin x \, dy + (3y^2 z^2 \sin x - x^4)\,dz$
   is an exact differential of some function ϕ and find this function.
   (ii) Show that $\mathbf{F} = (2xy + z^3)\,\mathbf{i} + x^2\,\mathbf{j} + 3xz^2\,\mathbf{k}$ is a conservative force field. Find the scalar potential. Find also the work done in moving an object in this field from $(1, -2, 1)$ to $(3, 1, 4)$.

4 (i) Show that the vector field

$$\mathbf{F} = (2xy^2 + yz)\,\mathbf{i} + (2x^2y + xz + 2yz^2)\,\mathbf{j} + (2y^2z + xy)\,\mathbf{k}$$

is conservative.

(ii) Show that $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is conservative and find ϕ such that $\mathbf{F} = \nabla\phi$.

5. (i) Show that $\mathbf{F} = (\sin y + z)\,\mathbf{i} + (x \cos y - z)\,\mathbf{j} + (x - y)\,\mathbf{k}$ is a conservative vector field and find a function ϕ such that $\mathbf{F} = \nabla\phi$.

(ii) Evaluate $\int_C 2xyz^2\, dx + (x^2z^2 + z\cos yz)\, dy + (2x^2yz + y\cos yz)\, dz$ where C is any path from $(0, 0, 1)$ to $(1, \pi/4, 2)$.

6. Show that the following vector functions $\mathbf{F}$ are irrotational and find the corresponding scalar ϕ such that $\mathbf{F} = \nabla\phi$.

(i) $\mathbf{F} = (\sin y + z\cos x)\,\mathbf{i} + (x\cos y + \sin z)\,\mathbf{j} + (y\cos z + \sin x)\,\mathbf{k}$.

(ii) $\mathbf{F} = (y\sin z - \sin x)\,\mathbf{i} + (x\sin z + 2yz)\,\mathbf{j} + (xy\cos z + y^2)\,\mathbf{k}$.

(iii) $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$.

7. Find a, b, c if $\mathbf{F} = (3x - 3y + az)\,\mathbf{i} + (bx + 2y - 4z)\,\mathbf{j} + (2x + cy + z)\,\mathbf{k}$ is irrotational.

8. Evaluate $\int_C yz\, dx + (xz + 1)\, dy + xy\, dz$, where C is any path from $(1, 0, 0)$ to $(2, 1, 4)$.

9. Show that the form under the integral sign is exact and evaluate

$$\int_{(0,2,1)}^{(2,0,1)} [ze^x\, dx + 3yz\, dy + (e^x + y^2)\, dz].$$

Answers 4

1. (i) Exact (ii) Exact

2. (i) Exact; $\phi = \dfrac{x^2 - y^2 - z^2}{2} + C$

(ii) Exact; $\phi = x + yz + C$

3. (i) $\phi = y^2z^3 \sin x - x^4z + C$

(ii) $\phi = x^2y - xz^3 + C$; 202

4. (ii) $\phi = \dfrac{1}{2}(x^2 + y^2 + z^2) + C.$

5. (i) $\phi = x\sin y + xz - yz + C.$

(ii) $\pi + 1$

6. (i) $\phi = x \sin y + z \sin x + y \sin z + C$

   (ii) $\phi = xy \sin z + \cos x + y^2 z + C$

   (iii) $\phi = \frac{1}{4}(x^4 + y^4 + z^4) + C$

7. $a = 2,\ b = -3, c = -4.$

8. 9

9. $e^2 - 5$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If C is the curve $x^2 + y^2 = 1, z = y^2$, then by Stoke's theorem $\oint_C (yz\, dx + zx\, dy + xy\, dz)$ is

   (a) 0 (b) 3

   (c) 5 (d) none of these

2. If S denotes the surface of the cube bounded by the planes $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0, z = a$ then by the application of Gauss divergence theorem the value of $\iint_S (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \bullet \mathbf{n}\, dS$ is

   (a) a^3 (b) $2a^3$

   (c) $3a^3$ (d) 0

3. The value of $\oint_C \mathbf{r} \bullet d\mathbf{r}$ is

   (a) 1 (b) 3

   (c) 0 (d) none of these

   (Kumaun 2007)

4. Gauss divergence theorem relates to

   (a) surface and volume integral (b) line and volume integral

   (c) line and surface integral (d) all of these **(Kanpur 2011)**

5. By Stoke's theorem the value of div curl **F** is

   (a) 1 (b) 2

   (c) 3 (d) 0 **(Kanpur 2013)**

6. By Green's theorem the value of $\oint_C (x^2 - \cosh y)\, dx + (y + \sin x)\, dy,$ where C is the rectangle with vertices $(0, 0), (\pi, 0), (\pi, 1), (0, 1)$, is
   (a) $1 - \cosh 1$ (b) $1 + \cosh 1$
   (c) $\cosh 1 - 1$ (d) $\cosh 1 + 1$

7. By Green's theorem the value of $\oint_C (\cos x \sin y - xy)\, dx + \sin x \cos y\, dy,$ where C is the circle $x^2 + y^2 = 1$, is
   (a) 1 (b) -1
   (c) 0 (d) none of these

8. For any closed surface $S, \iint_S \operatorname{curl} \mathbf{F} \bullet \mathbf{n}\, dS$ is
   (a) 0 (b) 1
   (c) -1 (d) 3

9. The value of $\iint_S \mathbf{r} \bullet \mathbf{n}\, dS,$ where S is a closed surface, is
   (a) V (b) $-V$
   (c) $3V$ (d) $-3V$

10. If $\mathbf{F} = ax\mathbf{i} + by\mathbf{j} + cz\,\mathbf{k}$; a, b, c are constants, then the value of $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$ is
   (a) $\frac{1}{3}\pi(a+b+c)$ (b) $-\frac{1}{3}\pi(a+b+c)$
   (c) $\frac{4}{3}\pi(a+b+c)$ (d) $-\frac{4}{3}\pi(a+b+c)$

11. The value of $\iint_S \mathbf{n}\, dS,$ for any closed surface S is
   (a) 0 (b) 1
   (c) -1 (d) 2

12. By Stoke's theorem the value of $\oint_C (e^x dx + 2y\, dy - dz),$ where C is the curve $x^2 + y^2 = 4, z = 2$, is
   (a) 1 (b) -1
   (c) 0 (d) 2

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. For any closed surface $S, \iint_S \operatorname{curl} \mathbf{F} \bullet \mathbf{n}\, dS = \ldots\ldots$

2. If $\mathbf{n}$ is the unit outward drawn normal to any closed surface S, then

$$\iiint_V \operatorname{div} \mathbf{n}\, dV = \ldots\ldots$$

(Bundelkhand 2008)

3. The value of $\oint_C \mathbf{r} \bullet d\,\mathbf{r} = \ldots\ldots$

(Agra 2008)

4. A necessary and sufficient condition that $\oint_C \mathbf{F} \bullet d\,\mathbf{r} = 0$ for every closed curve C lying in a simply connected region R is that $\nabla \times \mathbf{F} = \ldots\ldots$ identically.

5. By Stoke's theorem, $\oint_C \mathbf{F} \bullet d\,\mathbf{r} = \ldots\ldots$

6. $\iint_S \mathbf{r} \bullet \mathbf{n}\, dS = \ldots\ldots$

(Kumaun 2009)

True or False

Write 'T' for true and 'F' for false statement.

1. $\iint_S \mathbf{n}\, dS = \mathbf{0}$ for any surface S.
2. Green's theorem in plane is a special case of Stoke's theorem.
3. Green's theorem states that "the surface integral of the normal component of a vector **F** taken over a closed surface is equal to the integral of the divergence of **F** taken over the volume enclosed by the surface".

Answers

Multiple Choice Questions

1. (a) 2. (c) 3. (c) 4. (a) 5. (d)
6. (c) 7. (c) 8. (a) 9. (c) 10. (c)
11. (a) 12. (c)

Fill in the Blank(s)

1. 0 2. S 3. 0 4. 0
5. $\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS$ 6. $3V$

True or False

1. *F* 2. *T* 3. *F*

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